On Finding Short Resolution Refutations and Small Unsatisfiable Subsets

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Abstract. We consider the parameterized problems of whether a given set of clauses can be refuted within k resolution steps, and whether a given set of clauses contains an unsatisfiable subset of size at most k. We show that both problems are complete for the class W[1], the first level of the W-hierarchy of fixed-parameter intractable problems. Our results remain true if restricted to 3-SAT formulas and/or to various restricted versions of resolution including tree-like resolution, input resolution, and read-once resolution.
Applying a metatheorem of Frick and Grohe, we show that restricted to classes of locally bounded treewidth the considered problems are fixed-parameter tractable. Hence, the problems are fixed-parameter tractable for planar CNF formulas and CNF formulas of bounded genus, k-SAT formulas with bounded number of occurrences per variable, and CNF formulas of bounded treewidth.

1 Introduction

Resolution is a fundamental method for establishing the unsatisfiability of a given formula in Conjunctive Normal Form (CNF) using one single rule of inference, the *resolution rule*. This rule allows to infer the clause $C \cup D$ from clauses $C \cup \{x\}$ and $D \cup \{\neg x\}$. A CNF formula is unsatisfiable if and only if the empty clause can be derived by repeated application of the resolution rule. Resolution is easy to implement and provides the basis for many Automated Reasoning systems.

It is well known that certain unsatisfiable CNF formulas require an exponential number of resolution steps in order to be refuted [11]. Iwama [12] shows that, given a CNF formula F together with an integer k, deciding whether F has a resolution refutation with at most k steps is NP-complete. This result is strengthened by Alekhnovich et al. [2] by showing that the minimum number of resolution steps cannot be approximated within a constant factor, unless P = NP (this result also holds for stronger proof systems like Frege systems). A closely related question is the "automatizability" of resolution: is there an algorithm that finds a shortest resolution refutation R in polynomial time w.r.t.

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the number of steps in R? Alekhnovich and Razborov [3] show that resolution is not automatizable, assuming a parameterized intractability hypothesis regarding W[P]. For a survey of further results on the complexity of resolution, see, e.g., Beame and Pitassi [4] or Clote and Kranakis [6].

Parameterizing by the number of steps of a resolution refutation is of relevance if one has to deal with large CNF formulas which contain local inconsistencies. Evidently, one can use exhaustive search for finding a k-step resolution refutation of a CNF formula with n variables, yielding a time complexity of $n^{\mathcal{O}(k)}$. However, even if k is a small integer, say $k = 10$, exhaustive search becomes impractical for large n. The question rises whether one can find resolution refutations with a fixed number of steps significantly more efficient than by exhaustive search. The framework of parameterized complexity [8] offers a means for addressing this question. Here, problems are considered in two dimensions: one dimension is the usual size n of the instance, the second dimension is the parameter (usually a positive integer k). A parameterized problem is called *fixed-parameter tractable* (or *fpt*, for short) if it can be solved in time $f(k) \cdot n^{\mathcal{O}(1)}$ for some computable function f of the parameter. The parameterized complexity classes $\mathrm{W}[1] \subseteq \mathrm{W}[2] \subseteq \cdots \subseteq \mathrm{W}[\mathrm{P}]$ contain problems which are believed to be not fpt (see [8]); since all inclusions are believed to be proper, the hierarchy provides a means for determining the degree of parameterized intractability. A parameterized problem P *fpt reduces* to a parameterized problem Q if we can transform an instance (x, k) of P into an instance $(x', g(k))$ of Q in time $f(k) \cdot |x|^{\mathcal{O}(1)}$ (f, g are arbitrary computable functions), such that (x, k) is a yes-instance of P if and only if $(x', g(k))$ is a yes-instance of Q.

As a main result of this paper, we show that SHORT RESOLUTION REFUTATION, that is, refutability within k resolution steps, is complete for the class W[1]. We also show that this result holds true for several resolution refinements including tree-like resolution, regular resolution, and input-resolution. We establish the hardness part of the result by an fpt-reduction of the parameterized clique problem. As it appears to be difficult to establish W[1]-membership by reducing the problem to the canonical W[1]-complete problem on circuit satisfiability, we use results from descriptive parameterized complexity theory.

We show that refutability within k resolution steps can be expressed as a statement in positive (i.e., negation-free and $\forall$-free) first-order logic. This yields W[1]-membership as it was shown by Papadimitriou and Yannakakis [16] in the context of query evaluation over databases, that the evaluation of statements in positive first-order logic over finite structures is W[1]-complete.

Along these lines, we also show W[1]-completeness of SMALL UNSATISFIABLE SUBSET, that is, the problem of whether at most k clauses of a given CNF formula form an unsatisfiable formula. Furthermore, we pinpoint that all our W[1]-completeness results remain valid if the inputs are confined to 3-CNF formulas.

The notion of *bounded local treewidth* for classes of graphs (see Frick and Grohe [10]) generalizes several graph classes, like planar graphs, graphs of bounded treewidth, or graphs of bounded degree. By means of *incidence graphs* (see Section 2.1) we can apply this notion to classes of CNF formulas. Special cases are *planar CNF formulas* (CNF formulas with planar incidence graphs) and of (k,s)-*CNF formulas* (CNF formulas with k literals per clause and at most s occurrences per variable). Frick and Grohe [10] show that the evaluation of first-order statements over classes of graphs with locally bounded treewidth is fixed-parameter tractable (the result holds also for finite structures whose Gaifman graphs have locally bounded treewidth). Applying this powerful result, we obtain fixed-parameter tractability of SHORT RESOLUTION REFUTATION and SMALL UNSATISFIABLE SUBSET restricted to classes of CNF formulas with locally bounded treewidth. Thus the problems are tractable for planar CNF formulas and for (k,s)-CNF formulas.

Note that satisfiability is NP-complete for planar CNF formulas (Lichtenstein [15]) and $(3,4)$-CNF formulas (Tovey [18]), and even for the intersection of these two classes (Kratatochvíl [13]). However, satisfiability of CNF formulas of (globally) bounded treewidth is fixed-parameter tractable (Courcelle et al. [7], see also Szeider [17]).

2 Preliminaries and Notation

2.1 CNF Formulas

A *literal* is a propositional variable x or a negated variable $\neg x$; we also write $x^1 = x$ and $x^0 = \neg x$. A *clause* is a finite set of literals not containing a complementary pair $x, \neg x$. A *formula in conjunctive normal form* (or *CNF formula*, for short) F is a finite set of clauses. F is a k-*CNF formula* if the size of its clauses is at most k; F is a (k,s)-*CNF formula* if, additionally, every variable occurs in at most s clauses. The *length* of a CNF formula F is defined as $\sum_{C\in F} |C|$. For a CNF formula F, $var(F)$ denotes the set of variables x such that some clause of F contains x or $\neg x$. A literal x^ε is a *pure literal* of F if some clauses of F contain x^ε but no clause contains $x^{1-\varepsilon}$. F is *satisfiable* if there exists an assignment $\tau : var(F) \to \{0,1\}$ such that every clause of F contains some variable x with $\tau(x) = 1$ or some negated variable $\neg x$ with $\tau(x) = 0$; otherwise, F is called *unsatisfiable*. F is called *minimal unsatisfiable* if F is unsatisfiable and every proper subset of F is satisfiable. Note that minimal unsatisfiable CNF formulas have no pure literals. A proof of the following lemma can be found in Aharoni and Linial [1], attributed there to Tarsi.

Lemma 1 *A minimal unsatisfiable CNF formula has more clauses than variables.*

The *incidence graph* $I(F)$ of a CNF formula F is a bipartite graph; variables and clauses form the vertices of $I(F)$, a clause C and variable x are joined by an edge if and only if $x \in C$ or $\neg x \in C$ (see Fig. 1 for an example). A *planar CNF formula* is a CNF formula with a planar incidence graph.

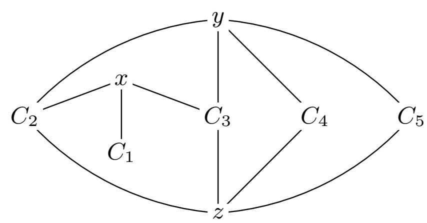

Fig. 1. The incidence graph $I(G)$ of the planar $(3,4)$-CNF formula $F = \{C_1, \ldots, C_5\}$ with $C_1 = \{x\}$, $C_2 = \{\neg x, y, z\}$, $C_3 = \{\neg x, y, \neg z\}$, $C_4 = \{\neg y, z\}$, $C_5 = \{\neg y, \neg z\}$.

2.2 Resolution

Let C_1, C_2 be clauses with $x \in C_1$, $\neg x \in C_2$, and $var(C_1) \cap var(C_2) = \{x\}$. The clause $C = (C_1 \cup C_2) \setminus \{x, \neg x\}$ is called the *resolvent* of C_1 and C_2. We also say that C is obtained by *resolving on* x, and we call C_1, C_2 *parent clauses* of C.

Recall that a vertex of a directed graph is called a *sink* if it has no successors, and it is called a *source* if it has no predecessors. A *resolution refutation* R is a directed acyclic graph whose vertices are labeled with clauses, such that

1. every non-source of R has exactly two predecessors and is labeled with the resolvent of the clauses labeling its predecessors;
2. R contains exactly one sink; the sink is labeled with the empty clause.

We call a non-source vertex of R a *step*. A clause labeling a source of R is called an *axiom* of R. R is a resolution refutation *of* a CNF formula F if all axioms of R are contained in F. It is well known that a CNF formula is unsatisfiable if and only if it has a resolution refutation (resolution is "refutationally complete").

In the sequel we will measure the size of resolution refutations in terms of the *number of steps*[1].

We refer to any decidable property of a resolution refutation as a *resolution refinement*. In particular, we will consider the following refinements:

– *Tree-like resolution*: The directed acyclic graph is a tree.
– *Regular resolution*: On any path from a source vertex to the sink, any variable is resolved at most once.
– *P-resolution*: at each resolution step, at least one of the parent clauses is a positive clause (i.e., a clause without negated variables);
– *Input resolution*: every vertex is either a source or has a predecessor which is a source.
– *Literal-once resolution*: distinct resolution steps resolve on distinct variables.
– *Read-once resolution*: distinct sources are labeled by distinct clauses.

[1] Another possible measure is the length of a refutation, defined as the total number of vertices (i.e., steps + source vertices). It is easy to verify that a resolution refutation with k steps has at most $k+1$ sources, and so its length is at most $2k+1$. Therefore, our results carry over if we bound the length instead of the number of steps.

Note that the first three refinements are refutationally complete, but the last three refinements are not. Note also that every literal-once resolution refutation is tree-like, read-once, and regular. Every input resolution refutation is tree-like.

2.3 Locally Bounded Treewidth

Treewidth, a popular parameter for graphs, was introduced by Robertson and Seymour in their series of papers on graph minors; see, e.g., Bodlaender's survey article [5] for definitions and references.

Let v be a vertex of a simple graph G and let r be some positive integer. $N_G^r(v)$ denotes the *r-neighborhood* of v, i.e., the set of vertices of G which can be reached from v by a path of length at most r. A class of graphs is said to have *locally bounded treewidth* if there exists a function f such that for all $r \geq 1$ and all vertices v of a graph G of that class, the treewidth of the subgraph included by $N_G^r(v)$ is at most $f(k)$. (Intuitively, the treewidth of the subgraph induced by an r-neighborhood of a vertex is a function of r and so less than the total number of vertices of G.) We give some examples of classes of graphs with locally bounded treewidth (see Frick and Grohe [10] for references).

- By trivial reasons, the class of graphs of treewidth $\leq t$ has locally bounded treewidth ($f(r) = t$).
- The class of planar graphs has locally bounded treewidth ($f(r) = 3r$); more generally, the class of graphs with genus $\leq g$ has locally bounded treewidth ($f(r) = \mathcal{O}(gr)$).
- The class of graphs with maximum degree $\leq d$ has locally bounded treewidth ($f(r) = d(d-1)^{r-1}$).

3 Statement of Main Results

Consider the following two parameterized problems.

> SHORT RESOLUTION REFUTATION
> *Input:* A CNF formula F.
> *Parameter:* A positive integer k.
> *Question:* Can F be refuted by at most k resolution steps? (i.e., can the empty clause be inferred from F by k applications of the resolution rule?).

> SMALL UNSATISFIABLE SUBSET
> *Input:* A CNF formula F.
> *Parameter:* A positive integer k.
> *Question:* Does F contain an unsatisfiable subset F' with at most k clauses?

Our main results are as follows.

Theorem 1 SHORT RESOLUTION REFUTATION *is* W[1]*-complete.*

The problem remains W[1]*-complete for the following resolution refinements: tree-like resolution, regular resolution, P-resolution, input resolution, read-once resolution, and literal-once resolution.*

Theorem 2 SMALL UNSATISFIABLE SUBSET *is* W[1]*-complete.*

Both theorems remain valid if inputs are confined to 3-CNF formulas.

We show fixed-parameter tractability for classes of CNF formulas whose incidence graphs have locally bounded treewidth:

Theorem 3 *For CNF formulas of locally bounded treewidth, the problems* SHORT RESOLUTION REFUTATION *and* SMALL UNSATISFIABLE SUBSET *are fixed-parameter tractable.*

Tractable cases include: planar CNF formulas, CNF formulas of bounded genus, and (k, s)-CNF formulas (k-CNF formulas with at most s occurrences per variable).

4 Proof of W[1]-hardness

We are going to reduce the following well-known W[1]-complete problem.

CLIQUE
Input: A graph G.
Parameter: A positive integer k.
Question: Is there a set $V' \subseteq V(G)$ of k vertices that induces a complete subgraph of G (i.e., a clique of size k)?

Given a simple graph $G = (V, E)$, $|V| = n$, and a positive integer k. We take distinct variables: x_i for $1 \le i \le k$, $y_{i,j}$ for $1 \le i < j \le k$, and $z_{v,i}$ for $v \in V$ and $1 \le i \le k$. We construct a CNF formula

$$F_G = \{C_{\text{start}}\} \cup F_{\text{edges}} \cup F_{\text{vertices}} \cup F_{\text{clean-up}}$$

where

$$\begin{aligned} C_{\text{start}} &= \{x_1, \dots, x_k\} \cup \{\, y_{i,j} : 1 \le i < j \le k \,\}, \\ F_{\text{edges}} &= \{\, \{\neg y_{i,j}, z_{u,i}, z_{v,j}\} : 1 \le i \le k,\ uv \in E \,\}, \\ F_{\text{vertices}} &= \{\, \{\neg x_i, z_{v,i}\} : 1 \le i \le k,\ v \in V \,\}, \\ F_{\text{clean-up}} &= \{\, \{\neg z_{v,i}\} : 1 \le i \le k,\ v \in V \,\}. \end{aligned}$$

We put

$$k' = \binom{k}{2} + 2k.$$

Lemma 2 *The following statements are equivalent.*

1. *F_G has an unsatisfiable subset F' with at most $k'+1$ clauses;*
2. *G contains a clique on k vertices;*
3. *F_G has a resolution refutation with at most k' steps which complies with the resolution refinements mentioned in Theorem 1;*
4. *F_G has a resolution refutation with at most k' steps.*

Proof. 1⇒2. We assume that F_G is unsatisfiable and choose a minimal unsatisfiable subset $F' \subseteq F_G$. First we show that

$$C_{\text{start}} \in F'. \tag{1}$$

Assume the contrary. Since F' has no pure literals, and since the variables x_i and $y_{i,j}$ occur positively only in C_{start}, we conclude that $F_{\text{vertices}} \cap F' = F_{\text{edges}} \cap F' = \emptyset$. Hence, in turn, $F_{\text{clean-up}} \cap F' = \emptyset$, thus $F' = \emptyset$. However, the empty formula is satisfiable, a contradiction. Thus C_{start} is indeed in F'. Since every clause in $F_{\text{edges}} \cup F_{\text{vertices}}$ contains the complement of exactly one variable of C_{start}, it follows that

$$|F_{\text{edges}} \cap F'| \geq \binom{k}{2}, \tag{2}$$

$$|F_{\text{vertices}} \cap F'| \geq k. \tag{3}$$

It also follows that for every $i \in \{1, \dots, k\}$ there is some $v \in V$ such that $z_{v,i} \in \mathit{var}(F_{\text{vertices}} \cap F')$. The latter implies

$$|F_{\text{clean-up}} \cap F'| \geq k. \tag{4}$$

Since $|F'| \leq k+1$ by assumption, (1) and the estimations (2)–(4) yield $|F'| = k'+1$. Hence the estimations (2)–(4) must be tight. Consequently, strengthening the above observation, we conclude that for every $i \in \{1, \dots, k\}$, there is *exactly* one vertex $v \in V$ such that $z_{v,i} \in \mathit{var}(F_{\text{vertices}} \cap F')$. Let $\varphi : \{1, \dots, k\} \to V$ be the map defined by

$$\varphi(i) = v \quad \text{if and only if} \quad z_{v,i} \in \mathit{var}(F_{\text{vertices}} \cap F').$$

In view of the tightness of the above estimations, we conclude that

$$\mathit{var}(F') = C_{\text{start}} \cup \{\, z_{\varphi(i),i} : 1 \leq i \leq k \,\}. \tag{5}$$

Consequently,

$$F_{\text{edges}} \cap F' = \{\, \{\neg y_{i,j}, z_{\varphi(i),i}, z_{\varphi(j),j}\} : 1 \leq i < j \leq k,\ \varphi(i)\varphi(j) \in E \,\}.$$

We conclude that the vertices $\varphi(1), \dots, \varphi(k)$ are mutually distinct; thus $\varphi(1), \dots, \varphi(k)$ induce a clique of size k in G.

2⇒3. Assume that G contains a clique on k vertices. Consequently, there is an injective map $\varphi : \{1, \dots, k\} \to V$ such that $\varphi(i)\varphi(j) \in E$ for all $1 \leq i < j \leq k$. We devise an input resolution refutation R of F_G, proceeding in three phases:

1. For $1 \leq i < j \leq k$ we resolve C_{start} with the clauses $\{\neg y_{i,j}, z_{\varphi(i),i}, z_{\varphi(j),j}\} \in F_{\text{edges}}$. We end up with the clause $C' = \{\, x_i, z_{\varphi(i),i} : i = 1, \ldots, k \,\}$.
2. For $1 \leq i \leq k$ we resolve C' with the clauses $\{\neg x_i, z_{\varphi(i),i}\} \in F_{\text{vertices}}$. We end up with the clause $C'' = \{\, z_{\varphi(i),i} : i = 1, \ldots, k \,\}$.
3. For $1 \leq i \leq k$ we resolve C'' with the clauses $\{\neg z_{\varphi(i),i}\} \in F_{\text{clean-up}}$. We end up with the empty clause.

By construction, R complies with the resolution refinements as claimed. Moreover, R contains $\binom{k}{2} + k + k = k'$ resolution steps.

3⇒4. Trivial.

4⇒1. Assume that F_G has a resolution refutation R with at most k' steps. Let F' denote the set of axioms of R. Note that F' is necessarily unsatisfiable, and since R has at most $k' + 1$ sources, $|F'| \leq k' + 1$ follows. □

The construction of F_G from F can be carried out in time $f(k)|E|^{\mathcal{O}(1)}$ for some function f. Thus Lemma 2 yields an fpt-reduction from CLIQUE to SHORT RESOLUTION REFUTATION with respect to the resolution refinements mentioned in Theorem 1, and an fpt-reduction from CLIQUE to SMALL UNSATISFIABLE SUBSET. Since CLIQUE is well-know to be W[1]-complete [8], we have established the hardness parts of Theorems 1 and 2.

4.1 3-CNF Formulas

Using a slight modification of the above construction, we can show that that the above hardness results hold for 3-CNF formulas. By means of a variant of Tseitin Extension [19], we transform a CNF formula F with clauses of size ≥ 2 into a 3-SAT formula $t(F)$, applying the following operations.

– Replace a clause $\{w_1, \ldots, w_n\}$ of size $n > 3$ by the clauses $\{w_1, w_2, u_1\}$, $\{\neg u_{n-3}, w_{n-1}, w_n\}$, and $\{\neg u_i, w_{i+2}, u_{i+1}\}$ for $i = 1, \ldots, n-4$ where u_i are new variables.
– Replace a clause $\{w_1, w_2\}$ by the clauses $\{w_1, w_2, u\}$, $\{\neg u, w_1, w_2\}$, u is a new variable.
– Replace a clause $\{w\}$ by the four clauses $\{w, u_1, u_2\}$, $\{w, u_1, \neg u_2\}$, $\{w, \neg u_1, u_3\}$, $\{w, \neg u_1, \neg u_3\}$, u_i are new variables.

It is straightforward that F is satisfiable if and only if $t(F)$ is satisfiable. Moreover, if F is minimal unsatisfiable, then so is $t(F)$, and the difference between the number of clauses and the number of variables remains the same for F and $t(F)$.

In view of the first part of the proof of Lemma 2 it follows that a minimal unsatisfiable subset F'' of $t(F_G)$ contains all $\binom{k}{2} + k - 2$ clauses of $t(\{C_{\text{start}}\})$, $\binom{k}{2}$ clauses of $t(F_{\text{edges}})$, $2k$ clauses of $t(F_{\text{vertices}})$, and $4k$ clauses of $t(F_{\text{clean-up}})$. In summary, the number of clauses in F'' is exactly

$$k'' = 2\binom{k}{2} + 7k - 2.$$

The proof of Lemma 2 carries over to $t(F_G)$ using k'' instead of k'.

5 Membership in W[1] and FPT Results

Let S denote a finite relational structure and φ a first-order (FO) formula (we quietly assume that the vocabularies of φ and S are compatible). S is a *model* of φ (in symbols $S \models \varphi$) if φ is true in S in the usual sense (see, e.g., [9,14] for further model theoretic definitions). Model-checking, the problem of deciding whether $S \models \varphi$, can be parameterized in different ways; in the sequel we will refer to the following setting.

FO MODEL CHECKING
Input: A finite structure S, a FO formula φ.
Parameter: The length of φ.
Question: Is S a model of φ?

Recall that a FO formula φ is *positive* if it does not contain negations or the universal quantifier $\forall$. We will use the following result of Papadimitriou and Yannakakis [16].

Theorem 4 FO MODEL CHECKING *for positive formulas is* W[1]*-complete.*

In [16] it is also shown that without the restriction to positive formulas, FO MODEL CHECKING is W[t]-hard for all t.

We associate to a relational structure S its *Gaifman graph* $G(S)$, whose vertices are the elements of the universe of S, and where two distinct vertices are joined by an edge if and only if they occur in the same tuple of some relation of S. By means of Gaifman graphs, one can speak of the treewidth of a relational structure and of classes of structures with locally bounded treewidth.

We shall use the following strong result of Frick and Grohe [10].

Theorem 5 FO MODEL CHECKING *for structures with locally bounded treewidth is fixed-parameter tractable.*

In the subsequent discussions, ρ denotes any of the resolution refinements mentioned in Theorem 1.

Let $y_1, y_2, \dots$ be an infinite supply of variables. For $k \geq 1$ we define the following classes of CNF formulas.

- $\mathcal{F}^k$ denotes the set of CNF formulas F with $var(F) = \{y_1, \dots, y_{k'}\}$ for some $k' \leq k$.
- $\mathcal{M}^k$ denotes the set of minimal unsatisfiable formulas in $\mathcal{F}^k$ with at most $k+1$ clauses.
- $\mathcal{R}^k$ denotes the set of CNF formulas $F \in \mathcal{F}^k$ such that F is the set of axioms of some resolution refutation with at most k steps; $\mathcal{R}^k_\rho$ is $\mathcal{R}^k$ restricted to ρ-resolution.

Lemma 3 *Every formula* $F \in \mathcal{R}^k$ *has at most* $k+1$ *clauses.*

Proof. We proceed by induction on k. If $k \leq 1$ then the lemma holds trivially, since either $F = \{\emptyset\}$ or $F = \{\{y_1\}, \{\neg y_1\}\}$. Assume that $k \geq 2$ and $F \in \mathcal{R}^k \setminus \mathcal{R}^{k-1}$. Consequently, there is a resolution refutation R with exactly k steps such that F is the set of axioms of R. We observe that R must contain a step v_0 where both predecessors v_1, v_2 of v_0 are sources. Let C_i denote the clause which labels v_i, $0 \leq i \leq 2$. We remove v_1 and v_2 from R and obtain a resolution refutation R' with $k-1$ steps. The vertex v_0 is now a source of R'. Let a and a' denote the number of axioms of R and R', respectively. Observe that a' is minimal if (1) C_0 is an axiom of R and (2) C_1, C_2 are not axioms of R'. Thus $a' \geq a - 2 + 1$. Since the set of axioms of R' belongs to $\mathcal{R}^{k-1}$, we have $a' \leq k$ by induction hypothesis, hence $|F| = a \leq k+1$ follows. □

Since there are less than 3^k clauses over the variables $\{y_1, \ldots, y_k\}$ (a variable appears positively, appears negatively, or does not appear in a clause), we conclude the following.

Lemma 4 *The sets $\mathcal{M}^k$ and $\mathcal{R}^k_\rho$ are finite and computable.*

We represent a CNF formula F by a relational structure $S_F = (P, N, V)$ as follows. For every variable x of F and every clause C of F, the universe of S_F contains distinct elements a_x and a_C, respectively. The relations of S_F are

$$P = \{\, (a_x, a_C) : x \in \mathit{var}(F),\ C \in F,\ x \in C \,\} \text{ (positive occurrence)},$$
$$N = \{\, (a_x, a_C) : x \in \mathit{var}(F),\ C \in F,\ \neg x \in C \,\} \text{ (negative occurrence)},$$
$$V = \{\, a_x : x \in \mathit{var}(F) \,\} \text{ (being a variable)}.$$

For example, the formula of Fig. 1 is represented by the structure $S_F = (P, N, V)$ with $P = \{(a_x, a_{C_1}),\ (a_y, a_{C_2}),\ (a_y, a_{C_3}),\ (a_z, a_{C_2}),\ (z, a_{C_4})\}$, $N = \{(a_x, a_{C_2}),\ (a_x, a_{C_3}),\ (a_y, a_{C_4}),\ (a_y, a_{C_5}),\ (a_z, a_{C_3}),\ (a_z, a_{C_5})\}$ and $V = \{a_x, a_y, a_z\}$.

In order to express that two variables are distinct without using negation, we also consider the structure $S_F^+ = (P, N, V, D)$ with the additional relation

$$D = \{\, (a_x, a_{x'}) : x, x' \in \mathit{var}(F),\ x \neq x' \,\} \text{ (distinctness)}.$$

The next lemma is a direct consequence of the definitions (cf. Fig. 1).

Lemma 5 *The incidence graph $I(F)$ and the Gaifman graph $G(S_F)$ are isomorphic for every CNF formula F.*

Let $k \geq 1$ and take two sequences of distinct FO variables $\vec{v} = v_1, \ldots, v_k$ and $\vec{w} = w_1, \ldots, w_{k+1}$. For a CNF formula $F \in \mathcal{F}^k$ with $F = \{C_1, \ldots, C_{k''}\}$, $k'' \leq k+1$, and $|\mathit{var}(F)| = k' \leq k$ we define the quantifier-free formula

$$\varphi[F] = \bigwedge_{1 \leq i < j \leq k'} \neg v_i = v_j \wedge \bigwedge_{j=1}^{k''} \left(\bigwedge_{y_i \in C_j} P(v_i, w_j) \wedge \bigwedge_{\neg y_i \in C_j} N(v_i, w_j) \right).$$

Furthermore, for $\mathcal{X}^k \in \{\mathcal{M}^k, \mathcal{R}^k_\rho\}$ we define

$$\varphi[\mathcal{X}^k] = \exists \vec{v}\ \exists \vec{w} \left(\bigwedge_{i=1}^{k} V(v_i) \wedge \bigvee_{F \in \mathcal{X}^k} \varphi[F] \right).$$

Similarly we define positive formulas $\varphi^+[F]$ using "$D(v_i, v_j)$" instead of "$\neg v_i = v_j$" and $\varphi^+[\mathcal{X}^k]$ using $\varphi^+[F]$ instead of $\varphi[F]$.

Lemma 6 *For every CNF formula F the following holds true.*

1. *F has a ρ-resolution refutation with at most k steps if and only if $S_F \models \varphi[\mathcal{R}^k_\rho]$ (i.e., $S^+_F \models \varphi^+[\mathcal{R}^k_\rho]$).*
2. *F contains an unsatisfiable subset of size at most $k+1$ if and only if $S_F \models \varphi[\mathcal{M}^k]$ (i.e., $S^+_F \models \varphi^+[\mathcal{M}^k]$).*

Proof. Let R be a ρ-resolution refutation of F with at most k steps, and let F' denote the set of axioms of R. Since all variables occurring in axioms of R are resolved in some of the resolution steps, $|var(F')| \leq k$ follows. We put $k' = |var(F')|$ and pick arbitrarily a bijection $r : var(F') \to \{y_1, \ldots, y_{k'}\}$. Renaming the variables in F' according to r yields a formula $r(F')$ which belongs to $\mathcal{R}^{k'}_\rho \subseteq \mathcal{R}^k_\rho$. It follows now from the definition of $\varphi[\mathcal{R}^k_\rho]$ that $S_F \models \varphi[\mathcal{R}^k_\rho]$ (equivalently, that $S^+_F \models \varphi^+[\mathcal{R}^k_\rho]$).

Now assume that F contains an unsatisfiable subset F' with at most $k+1$ clauses; we may assume that F' is minimal unsatisfiable. By Lemma 1 it follows that $|var(F')| \leq k$. Consequently, as in the previous case, we obtain from F' by renaming a formula $r(F') \in \mathcal{M}^k$, establishing $S_F \models \varphi[\mathcal{M}^k]$ and $S^+_F \models \varphi^+[\mathcal{M}^k]$.

The converse directions follow directly from the respective definitions of $\mathcal{R}^k_\rho$ and $\mathcal{M}^k$. ☐

To complete the proofs of Theorems 1, 2, and 3, it only remains to join together the above results: In view of Theorem 4, Lemma 6 implies directly the W[1]-membership part of Theorems 1 and 2. Whence Theorems 1 and 2 are shown true. Furthermore, Theorem 3 follows directly from Theorem 5 by Lemmas 5 and 6.

6 Concluding Remarks

Numerous parameterized problems have been identified as being W[1]-complete, for example, the Halting Problem for nondeterministic Turing machines, parameterized by the number of computation steps. Our Theorem 1 links parameterized complexity with the length of resolution refutations, another fundamental concept of Logic and Computer Science; thus our result provides additional evidence for the significance of the class W[1].

Our positive results, the fp-tractability of SHORT RESOLUTION REFUTATION and SMALL UNSATISFIABLE SUBSET for classes of CNF formulas of locally bounded tree-width, are obtained by application of Frick and Grohe's metatheorem which does not provide practicable algorithms. However, the results show that fp-tractability can be achieved in principle, and so that further efforts for finding more practicable algorithms based on the particular combinatorics of the problems are encouraged. We think that the classes of planar CNF formulas and (k, s)-CNF formulas are good candidates for such an approach.

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