Parameterized Low-distortion Embeddings - Graph metrics into lines and trees

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Abstract

We revisit the issue of low-distortion embedding of metric spaces into the line, and more generally, into the shortest path metric of trees, from the parameterized complexity perspective. Low-distortion embeddings of a metric space into the line, or into some other "simple" metric space, (that is, a mapping that preserves distances, up to some small multiplicative factor called the *distortion*) have many applications in computer science.

Let $M = M(G)$ be the shortest path metric of an unweighted graph $G = (V, E)$ on n vertices. We describe algorithms for the problem of finding a low distortion non-contracting embedding of M into line and tree metrics.

- Our first result is that the problem of embedding M into the line, parameterized by the distortion d, is fixed parameter tractable (FPT). We describe an algorithm that on input (G, d) either constructs an embedding of $M(G)$ into the line with distortion at most d, or concludes that no such embedding exists. The running time of our algorithm is $O(nd^4(2d+1)^{2d})$ which is linear for every fixed d and polynomial for $d = O(\lg n/\lg\lg n)$. This is a significant improvement over the best previous algorithm, of Bădoiu *et al.* [SODA 2005] that has a running time of $O(n^{4d+2}d^{O(1)})$.
- We generalize the result on embedding into the line by proving that for any tree T with maximum degree Δ, embedding of M into a shortest path metric of T is FPT, parameterized by (Δ, d). This result can also be viewed as a generalization (albeit with a worse running time) of the previous FPT algorithm due to Kenyon, Rabani and Sinclair [STOC 2004] that was limited to the situation where $|G| = |T|$.

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1 Introduction

A mapping (or embedding) f between two metric spaces M and M' with distance functions D and D' has *contraction* c_f and *expansion* e_f if for every pair of points p, q in M, $D(p,q) \leq D'(f(p), f(q)) \cdot c_f$ and $D(p,q) \cdot e_f \geq D'(f(p), f(q))$ respectively. We say that f is *non-contracting* if c_f is at most 1. A non-contracting mapping f has *distortion* d if e_f is at most d. The problem of finding a low-distortion embedding between metric spaces is a fundamental mathematical problem with many applications in computer science [12, 14], and has been studied intensively.

Many of the known results on low-distortion embeddings are combinatorial (or "absolute"), that is, of the form: any metric M from a given class can be embedded into a metric M' with a low distortion. However, in many situations, even for very simple metrics, the worst-case distortion is too large to be useful. For example, the embedding of the shortest path metric of a star on n vertices into the line metric requires distortion $\Omega(n)$. Applications often require algorithms for *finding* low distortion embeddings, and the study of algorithmic issues of metric embeddings has recently begun to develop [1, 2, 3, 13]. Because in many applications the distortion needs to be small, it is natural to study the algorithmic issues in the framework of parameterized complexity [7, 10, 15].

Definition 1.1. *For an (unweighted) graph G, we define the metric $M(G)$ as the pair $(V(G), D)$, where $V(G)$ is the vertex set of G and $D(u,v)$ is the shortest path distance between u and v in G. For a subset S of $V(G)$, we say that $M[S] = (S, D')$ (where D' is D restricted to S^2) is the submetric of $M(G)$ induced by S. When we refer to an embedding of a graph G or a subset S of vertices of G we in fact mean an embedding of $M(G)$ and $M(G)[S]$ respectively.*

What would one expect about the complexity of embedding a graph minimum distance metric $M = M(G)$ into the line, parameterized by the distortion d? At first glance, the problem seems to closely resemble the BANDWIDTH MINIMIZATION problem, which is hard for $W[t]$ for all t [4, 5]. The best previous algorithm (by Bădoiu *et al.* [2]) has a running time where d appears in the exponent of n, the number of vertices of G. Surprisingly, however, our first main result shows that this fundamental problem admits an FPT algorithm. Our second main result extends this to the problem of embedding M into a tree metric, and complements an earlier FPT result of Kenyon, Rabani and Sinclair [13] for bijective embeddings.

We hope that these positive results on basic metric embedding problems will open up the area to much further investigation from the parameterized point of view.

Related work. Bădoiu *et al.* [1, 2, 3] investigated approximation and exact algorithms for embedding unweighted and weighted graph metrics into line and trees metrics. In particular, in [2], they obtained an algorithm that finds an embedding of the shortest path metric space $M = M(G)$ of an unweighted graph G on n vertices with distortion bounded by d (if one exists) in time $n^{O(d)}$. Bădoiu *et al.* [1, 3] describe approximation algorithms and hardness results for embedding general metrics into the line and trees metrics respectively. For example, they show that the minimum distortion for a line embedding is hard to approximate up to a factor polynomial in n even for weighted trees with polynomial spread. (the ratio of maximum/minimum weights).

Kenyon *et al.* [13] initiated the study of metric bijective embeddings. In particular, they provided fixed parameter tractable algorithms for the bijective embedding of unweighted graph metrics into the metric of a tree with bounded maximum degree Δ. The running time of their algorithm is $n^2 \cdot 2^{\Delta^{\alpha^3}}$ where α is the maximum of c_f and e_f. An important point, observed in [2], is that constraining the embedding to be bijective (not just injective, as in our case) is crucial for the correctness of the algorithms from [13]. Hall and Papadimitriou [11] studied the hardness of approximation for bijective embeddings.

Low-distortion embedding into a *spanning tree* of a graph is known as the minimum-stretch spanning tree problem [9, 8, 16]. Cai and Corneil [6] have shown that, to decide whether a graph G has a tree spanner with stretch t is NP-complete for any fixed $t \geq 4$ and is linear time solvable for $t = 1, 2$ (the status of the case $t = 3$ is open).

2 Embedding Into Line

In this section we give an algorithm that for a given a graph G on n vertices and integer d, decides whether G can be embedded into the line with distortion at most d in time $O(nd^4(2d+1)^{2d})$. Before we proceed to the details of the algorithm we need a few observations that allows us to only consider a specific kind of embeddings.

For a non-contracting embedding f of a graph G into the line, we say that vertex u *pushes* vertex v if $D(u,v) = |f(u) - f(v)|$.

Observation 2.1. *If $f(u) < f(v) < f(w)$ and u pushes w, then u pushes v and v pushes w.*

Proof. By the triangle inequality, $D(u,w) \leq D(u,v) + D(v,w)$. Since u pushes w and because f is non-contracting, we have that $D(u,w) = f(w) - f(u) = (f(w) - f(v)) + (f(v) - f(u)) \geq D(u,v) + D(v,w)$. Thus $(f(w) - f(v)) + (f(v) - f(u)) = D(u,v) + D(v,w)$, which yields that $f(w) - f(v) = D(v,w)$ and $f(v) - f(u) = D(u,v)$. □

For an embedding f, let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices such that $f(v_1) < f(v_2) < \cdots < f(v_n)$. We say that f is *pushing* if v_i pushes v_{i+1}, for each $1 \leq i \leq n-1$.

Observation 2.2. *If G can be embedded into the line with distortion d, then there is a pushing embedding of G into the line with distortion d. Furthermore, every pushing embedding of G into the line is non-contracting.*

Proof. Among all embeddings of G into the line with distortion d, let us choose f such that

$$\sum_{2 \leq i \leq n} |f(v_i) - f(v_{i-1})| \text{ is minimized.}$$

We claim that f is pushing. Indeed, if f is not pushing, then there is a minimum integer $q \geq 1$ such that v_q does not push v_{q+1}. By Observation 2.1, for every $p \leq q$ and $r \geq q+1$, $D(v_p, v_r) > f(v_r) - f(v_p)$. But then embedding f', $f'(v_i) = f(v_i)$ for $i \leq q$ and $f'(v_i) = f(v_i) - 1$ for $i > q$, is non-contracting embedding of distortion d, which is a contradiction to the choice of f.

To prove that every pushing embedding of G into the line is non-contracting, we observe that for each $b > a \geq 1$, $f(v_b) - f(v_a) = \sum_{i=a+1}^{b} f(v_i) - f(v_{i-1}) = \sum_{i=a+1}^{b} D(v_i, v_{i-1}) \geq D(v_a, v_b)$. □

Observation 2.3. *Let f be a pushing embedding of a connected graph G into the line with distortion at most d. Then $D(v_{i-1}, v_i) \leq d$ for every $1 \leq i \leq n$.*

Proof. Suppose for contradiction that $D(v_{i-1}, v_i) > d$ for some i. As the endpoints of any edge can be mapped at most d apart, this implies that G is disconnected. □

By Observation 2.2, it is sufficient to work only with pushing embeddings. Our algorithm is based on dynamic programming over small intervals of the line. The intuition behind the algorithm is as follows. Let us consider a distortion d embedding of G into the line and an interval of length $2d+1$ of the line. First, observe that no edge can have one end-point to the left of this interval and one end-point to the right. This means that if there is a vertex u embedded to the left of this interval and another vertex v that has been embedded to the right, then the set of vertices embedded into the interval form an u, v-separator. Moreover, for each edge, its end-points can be mapped at most d apart of each other, and hence there is no edge with one end-point to the left of this interval and the other end-point in the rightmost part of this interval. Thus just by looking at the vertices mapped into an interval of length $2d+1$, we deduce which of the remaining vertices of G were mapped to the left and which were mapped to the right of this interval. This is a natural division of the problem into independent subproblems and the solutions to these subproblems can be used to find an embedding of G. Next we formalize this intuition by defining *partial embeddings* and showing how they are glued onto each other to form a distortion d embedding of the input graph.

It is well known (and it follows from Observation 2.2) that there always exists an optimal embedding with all the vertices embedded into *integer coordinates* of the line. Without loss of generality, in the rest of this section we only consider pushing embeddings of this type. We also assume that our input graph G is *connected*.

Definition 2.4. *For a graph G and a subset $S \subseteq V(G)$, a* partial embedding *of S is a function $f : S \to \{-(d+1), \ldots, d+1\}$. We define $S_f^{[a,b]}$, $-(d+1) \leq a \leq b \leq d+1$, to be the set of all vertices of S which are mapped into $\{a, \cdots, b\}$ by f (let us remark that this can be $\emptyset$). We also define $S_f^L = S_f^{[-(d+1),-1]}$ and $S_f^R = S_f^{[1,d+1]}$. For an integer x, $-(d+1) \leq x \leq d+1$, we put $S_f^x = S_f^{[x,x]}$. Finally, we put $L(f)$ ($R(f)$) to denote the union of the vertex sets of all connected components of $G \setminus S$ that have neighbors in S_f^L (S_f^R).*

Definition 2.5. *A partial embedding f of a subset $S \subseteq V(G)$ is called feasible if*

1. *f is a non-contracting distortion d embedding of S;*
2. *$L(f) \cap R(f) = \emptyset$;*
3. *Every neighbor of S_f^0 is in S;*
4. *if $R(f) = \emptyset$, then S_f^{d+1} is nonempty;*
5. *if $L(f) = \emptyset$, then $S_f^{-(d+1)}$ is nonempty;*
6. *if $f(u) + 1 < f(v)$ and $S_f^{[f(u)+1, f(v)-1]} = \emptyset$, then $f(v) - f(u) = D(u, v)$. (In other words, u pushes v.)*

The properties 1, 2, and 3 of this definition will be used to show that every distortion d embedding of G into the line can be described as a sequence of feasible partial embeddings that have been glued onto each other. Properties 4, 5 and 6 are helpful to bound the number of feasible partial embeddings.

Definition 2.6. *Let f and g be feasible partial embeddings of a graph G, with domains S_f and S_g, respectively. We say that g* succeeds *f if*

1. $S_f^{[-d,d+1]} = S_g^{[-(d+1),d]} = S_f \cap S_g$;
2. *for every* $u \in S_f \cap S_g$, $f(u) = g(u) + 1$;
3. $S_g^{d+1} \subseteq R(f)$;
4. $S_f^{-(d+1)} \subseteq L(g)$.

The properties 1 and 2 describe how one can glue a partial embedding g that has been shifted one to the right onto another partial embedding f. Properties 3 and 4 are employed to enforce "intuitive" behaviour of the sets $L(f)$, $R(f)$, $L(g)$ and $R(g)$. That is, since g is glued on the right side of f, everything to the right of g should appear in the right side of f. Similarly, everything to the left of f should be to the left of g.

Lemma 2.7. *For every pair of two feasible partial embeddings f and g of subsets S_f and S_g of $V(G)$ such that g succeeds f, we have $R(f) = R(g) \cup S_g^{d+1}$ and $L(g) = L(f) \cup S_f^{-(d+1)}$.*

Proof. Let us prove that $R(f) = R(g) \cup S_g^{d+1}$. (The proof of $L(g) = L(f) \cup S_f^{-(d+1)}$ is similar.) Because g succeeds f, we have that $S_g^{d+1} \subseteq R(f)$. Let C be the vertex set of a connected component of $G \setminus S_g$ such that $C \subseteq R(g)$. As $S_f^{-(d+1)} \subseteq L(g)$, the subgraph $G[C]$ induced by C, is a connected component of $G \setminus (S_f \cup S_g)$. If C contains a neighbor of S_g^{d+1}, then $C \subseteq R(f)$ (this is because $S_g^{d+1} \subseteq R(f)$ and C and S_g^{d+1} are in the same connected component of $G \setminus S_f$). On the other hand, if C contains no neighbor of $S_g^{(d+1)}$, then, as $C \subseteq R(g)$, C has a neighbor in $S_g^{(1,d)} \subseteq S_f^R$. Therefore $C \subseteq R(f)$, which in turn implies that $R(g) \subseteq R(f)$. Thus, we have proved that $R(g) \cup S_g^{d+1} \subseteq R(f)$.

Let us now show that $R(f) \subseteq R(g) \cup S_g^{d+1}$. Let C be the vertex set of a connected component of $G \setminus S_f$ such that $C \subseteq R(f)$. C contains no neighbors of $S_f^{-(d+1)}$ thus C is a connected component of $G \setminus S_g^{(-(d+1),d)}$. If C does not contain S_g^{d+1}, then C is a connected component of $G \setminus S_g$. Furthermore, as $C \subseteq R(f)$, C has a neighbor in $S_f^{(1,d+1)} \subseteq S_g^{(0,d)}$. As S_g^0 has no neighbors outside of S_g, C has a neighbor in S_g^R implying $C \subseteq R(g)$. On the other hand, if C contains S_g^{d+1}, then every connected component C' of $G[C] \setminus S_g^{d+1}$ is a connected component of $G \setminus S_g$ that has a neighbor in $S_g^{d+1} \subseteq S_g^R$. This concludes the proof that $R(f) \subseteq R(g) \cup S_g^d$, implying $R(f) = R(g) \cup S_g^{d+1}$. □

Theorem 2.8. *Let G be a graph. For every integer d, G has an embedding of distortion at most d if and only if there exists a sequence of feasible partial embeddings $f_0, f_1, f_2, \dots, f_t$ such that for each $0 \le i \le t-1$, f_{i+1} succeeds f_i, and $L(f_0) = R(f_t) = \emptyset$.*

Proof. Let f be a pushing embedding of G with distortion d which maps all vertices to integers greater than or equal to $-(d+1)$ and maps one vertex to $-(d+1)$. Let t be the smallest integer such that $f(v) \le t + d + 1$ for every $v \in V$. For every $0 \le i \le t$, let S_i be the set of vertices that f maps to $\{i - (d+1), \dots, i + d + 1\}$. We define $f_i : S_i \to \{-(d+1), \dots, d+1\}$ to be $f_i(v) = f(v) - i$, $v \in S_i$. Then for every $i \le t-1$, f_i is a feasible partial embedding, f_{i+1} succeeds f_i, and $L(f_0) = R(f_t) = \emptyset$.

In the other direction, let $f_0, f_1, f_2, \dots, f_t$ be a sequence of feasible partial embeddings such that for each i, f_{i+1} succeeds f_i and $L(f_0) = R(f_t) = \emptyset$. Let S_i be the domain of f_i. First we show that for every vertex v there is i such that $v \in S_i$. If $v \notin S_0$, then $v \in R(f_0)$. Let k be the largest integer such that $v \in R(f_k)$. Because $R(f_t) = \emptyset$, we have that $k < t$. Thus, $v \in R(f_k) \setminus R(f_{k+1})$. By Lemma 2.7, $R(f_k) \setminus R(f_{k+1}) \subseteq S_{f_{k+1}}^{d+1}$ which implies that $v \in S_{k+1}$.

We claim that for every $v \in S_i \cap S_j$, $f_i(v)+i = f_j(v)+j$. Indeed, let k be the smallest integer such that $v \in S_k$. Let $k' = \min\{t, f_k(v)+k+d+1\}$. For every i and j, such that $k \le i, j \le k'$, we have $f_i(v)+i = f_j(v)+j$. Furthermore, if $k' < t$, then $v \in L(f_{k'+1})$ and thus by Lemma 2.7, $v \in L(f_{k''})$ for every $k' < k'' \le t$. Since k is the smallest integer such that $v \in S_k$, we have that if $v \in S_i \cap S_j$, then $f_i(v)+i = f_j(v)+j$.

¿From the previous two paragraphs, we conclude that there is a function f such that for every $v \in S_i$, $f(v) = f_i(v)+i$. It remains to prove that f is a distortion d embedding of G into the line. We say that a pair of vertices u and v are in conflict if either $|f(u)-f(v)| < D(u,v)$, or if $|f(u)-f(v)| > d \cdot D(u,v)$. Let us note that if no pair of vertices are in conflict, then f is a distortion d embedding of G. We prove that no two vertices in $S_i \cup L(f_i)$ are in conflict by induction on i. For $i=0$ this is true as f_0 is a feasible partial embedding. Assume now that the statement is true for every $i < k$.

If $S^{d+1}_{f_k}$ is empty, then the statement trivially holds for k. Otherwise, for some vertex v, $S^{d+1}_{f_k} = \{v\}$. To complete the proof, it is sufficient to show that v is not in conflict with any other vertex u in $S_k \cup L(f_k)$. If u is in S_k, u and v are not in conflict because f_k is a feasible partial embedding. If u is not in S_k, then u is in $L(f_k)$ and every shortest path from u to v in G must contain a vertex $w \in S^L_k$. Since $f(u) \le f(w) \le f(v)$, we have that $|f(v)-f(u)| = f(v)-f(w)+f(w)-f(u) \ge D(v,w)+D(w,u) = D(v,u)$. Therefore, $|f(v)-f(u)| = f(v)-f(w)+f(w)-f(u) \le d \cdot D(v,w) + d \cdot D(w,u) = d \cdot D(v,u)$. Thus no pair of vertices in $S_i \cup L(f_i)$ are in conflict for every $i \le t$. However, for $i = t$, $S_i \cup L(f_i) = V(G)$ and we conclude that no pair of vertices are in conflict. □

For a vertex v of a graph G and integer $r \ge 0$ we denote the ball of radius r centered in v, which is the set of vertices at distance at most r in G, by $B(v,r)$. The *local density* of a graph G is $\delta = \max_{v \in V(G), r>0} \frac{|B(v,r)-1|}{2r}$. We will apply the following well known lower bound on distortion.

Lemma 2.9. *[1] [Local Density] Let G be a graph that can be embedded into the line with distortion d. Then d is at least the local density δ of G.*

Applying Lemma 2.9 we can bound the number of possible feasible partial embeddings. Observe that each feasible partial embedding f can be represented as a number $1 \le t \le d$ and a sequence of vertices $v_0v_1\cdots v_q$ such that $t + \sum_{i=1}^{q} D(v_{i-1}, v_i) \le 2d+1$ and $D(v_{i-1}, v_i) \le d$ for every $i \ge 1$. This is done by simply saying that the domain S of f is the set $\{v_0, v_1, \cdots v_q\}$ and that $f(v_a) = -(d+1) + t + \sum_{i=1}^{a} D(v_{i-1}, v_i)$ Let $\mathcal{N}(x)$ be the maximum number of sequences $v_0v_1\cdots v_q$ such that $\sum_{i=1}^{q} D(v_{i-1}, v_i) = x$ and $D(v_{i-1}, v_i) \le d$ for every $i \ge 1$, where maximum is taken over all $v_0 \in V(G)$. For any negative number x, $\mathcal{N}(x) = 0$.

Lemma 2.10. *For $x \in \mathbb{Z}$, $\mathcal{N}(x) \le (2d+1)^x$.*

Proof. We prove Lemma by induction on x. For $x \le 0$, the statement is trivially true. Suppose that the inequality holds for every $x' < x$. For a vertex v_0, let $\mathcal{S}$ be the set of all vertex sequences $v_0v_1\cdots v_q$ starting with v_0 with the property that $\sum_{i=1}^{q} D(v_{i-1}, v_i) = x$. For $i \in \{1, \ldots, d\}$, let $\mathcal{S}_i$ be the set of sequences in $\mathcal{S}$ such that $D(v_0, v_1) = i$. Let $C(v_0, i) = |B(v_0, i) \setminus B(v_0, i-1)|$. Then $|\mathcal{S}_i| \le C(v_0, i) \cdot \mathcal{N}(x-i)$ and $|\mathcal{S}| = \sum_{i=1}^{d} |\mathcal{S}_i| \le \sum_{i=1}^{d} C(v_0, i) \cdot \mathcal{N}(x-i)$. By the induction assumption, $\sum_{i=1}^{d} C(v_0, i) \cdot \mathcal{N}(x-i) \le \sum_{i=1}^{d} C(v_0, i) \cdot (2d+1)^{x-i}$. Furthermore, by Lemma 2.9, we have that $\sum_{j=1}^{i} C(v_0, i) \le 2di$ for every i. Because $(2d+1)^y$ is a convex function of y, it follows that the sum $\sum_{i=1}^{d} C(v_0, i) \cdot (2d+1)^{x-i}$

subject to the constraints $\sum_{j=1}^{i} C(v_0, i) \leq 2di$ is maximized when each of $C(v_0, i) = 2d$. In this case $\sum_{i=1}^{d} C(v_0, i) \cdot (2d+1)^{x-i} \leq 2d \cdot \sum_{i=1}^{d} (2d+1)^{x-i}$ which is a geometric sequence with sum $2d \cdot (2d+1)^{x-d} \cdot \frac{(2d+1)^d - 1}{2d} < (2d+1)^x$. Since this holds for each choice of v_0, we conclude that the inequality holds also for x. □

Corollary 2.11. *For a graph G with local density at most d the number of possible feasible partial embeddings of subsets of $V(G)$ is at most $O(n(2d+1)^{2d})$.*

Proof. By discussions preceding Lemma 2.10, for each fixed first vertex v_0 and each value of t, there are at most $\mathcal{N}(2d+1-t)$ feasible partial embeddings that map v_0 to $-(d+1)+t$ Thus the number of feasible partial embeddings is at most $\sum_{t=1}^{d} n\mathcal{N}(2d+1-t)$. By Lemma 2.10, this is at most $n \cdot \sum_{t=1}^{d} (2d+1)^{2d+1-t} \leq \frac{3}{2} n(2d+1)^{2d}$. □

Now we are in the position to state the main result of this section.

Theorem 2.12. *There is an algorithm that decides whether a graph G can be embedded into the real line with distortion at most d in time $O(nd^4(2d+1)^{2d})$.*

Proof. The algorithm proceeds as follows. First, check whether G has local density δ bounded by d. Checking the local density of G can be done in time linear in n because if $|E(G)| \geq nd$ we can immediately answer "no". If $\delta > d$, answer "no". Otherwise, we can test whether the conditions of theorem 2.8 apply. That is, we construct a directed graph $\mathcal{D}$ where the vertices are feasible partial embeddings and there is an edge from a partial embedding f_x to a partial embedding f_y if f_y succeeds f_x. Checking the conditions of Theorem 2.8, reduces to checking for the existence of a directed path starting in a feasible partial embedding f_0 with $L(f_0) = \emptyset$ and ending in a feasible partial embedding f_t with $R(f_t) = \emptyset$. This can be done in linear time in the size of $\mathcal{D}$ by running a depth first search in $\mathcal{D}$. The number of vertices in $\mathcal{D}$ is at most $O(n(2d+1)^{2d})$. Every vertex of $\mathcal{D}$ has at most $O(d^2)$ edges going out of it, as a feasible partial embedding f_y succeeding another feasible partial embedding f_x is completely determined by f_x together with the vertex that f_y maps to $d+1$ (or the fact that f_y does not map anything there). Using prefix tree like data structures one can test whether a given partial embedding f_x succeeds another in $O(d^2)$ time. The total running time is then bounded by $O(nd^4(2d+1)^{2d})$. □

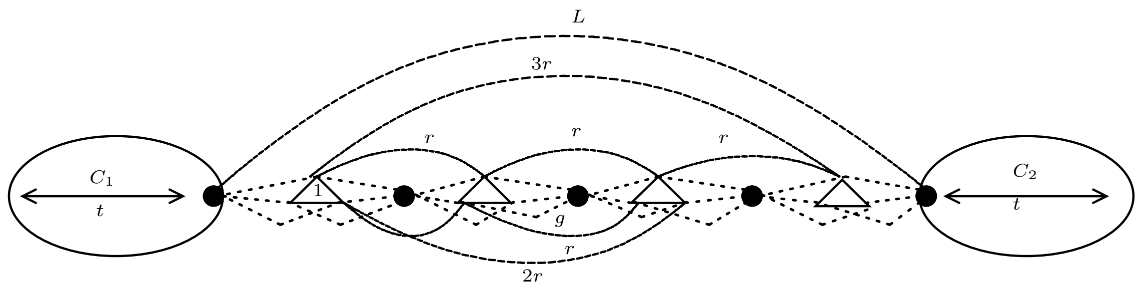
3 Embedding General Metric Into Line

3.1 Weighted Graph Metrics With Small Edge Weights into Line

3.2 General Metrics into Line is Hard for Fixed Rational $d \geq 2$

We show that deciding whether a given general metric can be embedded into the line with distortion at most d is NP-complete for every fixed rational $d \geq 2$. Our reduction is from 3-coloring, one of the classical NP-complete problems. On input $G = (V, E)$ to 3-coloring we show how to construct an edge weighted graph $G' = (V', E')$. For an edge $uv \in E'$, $w(uv)$ will be the weight if the edge uv. The weighted shortest path metric $M(G')$ will then be the input to our embedding problem. Let $n = |V|$, $m = |E|$ and $d = \frac{a}{b} \geq 2$ for some integers a and b. We order the edges of G into $e_1 \cdots e_m$ and chose the integers $g = 5a-1$, $r = 10b$, $q = m(2n+1)$, $L = 10qb$ and $t = abL+1$. We start constructing G' by making two cliques C_1 and C_2 of size t. Let $C_1 = \{c_1, c_2, \cdots c_t\}$ and $C_2 = \{c'_1, c'_2, \cdots c'_t\}$. Let $w(c_i c_j) = w(c'_i c'_j) = \lceil |i-j|/d \rceil$. Now, we make $q-1$ *separator vertices* and label them

$s_1 \cdots s_{q-1}$. We make q gadgets $T_1 \cdots T_q$ encoding the edges of G. For every edge $e_i = uv$ there are $2n+1$ gadgets, namely T_{i+mp} for every $0 \le p < 2n+1$. Each such gadget, say T_{i+mp}, consists of three vertices, one vertex *corresponding* to u, one vertex corresponding to v and one vertex corresponding to e_i. These three vertices form a triangle with edges of weight 1. For every j between 1 and q we connect all vertices of T_j to s_{j-1} and s_j with edges of weight g. Whenever this implies that we need to connect something to the non-existing vertices s_0 and s_q we connect to c_t and c'_1 respectively. Now, for every pair of vertices $x \in T_i$ and $y \in T_j$ that correspond to the same vertex or edge of G we add an edge of weight $r|i-j|$ between x and y. Finally, we add an edge with weight L between c_t and c'_1. This concludes the construction of G'.



Lemma 3.1. *For every edge uv in E', $D_{G'}(u,v) = w(uv)$.*

Proof. $D_{G'}(u,v) \le w(uv)$ for every edge uv so it is sufficient to prove $D_{G'}(u,v) \ge w(uv)$. If $w(uv) = 1$ then $D_{G'}(u,v) \ge w(uv)$, so suppose $w(uv) > 1$. In this case uv either has both endpoints in C_1 or C_2, is the edge $c_t c'_1$, is an edge from c_t to a vertex in T_1, an edge from c'_1 to a vertex in T_q, an edge incident to a separator vertex or an edge between a vertex in a gadget T_i and a vertex in a gadget T_j. If both u and v lie inside C_1 every shortest $u-v$ path lies entirely within C_1. For every w in C_1 we have that $w(u,v) \le w(u,w)+w(w,v)$ so $w(uv) \le D_{G'}(u,v)$. Similarly, if both u and v lie inside C_2 then $w(uv) \le D_{G'}(u,v)$. If uv is incident to a separator vertex then $w(uv) = g$ and $D_{G'}(uv) \ge g$ because every separator vertex is only incident to edges with weight g. If uv is an edge from c_t to a vertex in T_1 or an edge from c'_1 to a vertex in T_q then $w(uv) = g$ and $D_{G'}(uv) \ge g$ because every edge with one endpoint inside $C_1 \cup C_2$ and one endpoint outside of $C_1 \cup C_2$ has weight exactly g.

Now, if uv is an edge between a vertex in a gadget T_i and a vertex in a gadget T_j, $w(uv) = r|i-j|$. Observe that a path from u to v with length smaller than $r|i-j|$ can never use the edge $c_t c'_1$ and thus will never visit the set $C_1 \cup C_2$. We prove that the distance between a u and v is at least $r|i-j|$ by induction on $|i-j|$. If $|i-j| = 1$ then any path containing an egde with one endpoint in $T_{i'}$ and another in $T_{j'}$ with $i' \ne j'$ will have length at least r. Any path from u to v that does not contain any such edges must contain at least one separator vertex as an in intermediate vertex and thus have length at least $2g = 20a - 2 > 10b = r$. We now suppose that the induction hypothesis is true whenever $|i-j| < z$ and show that it also must hold when $|i-j| = z$. If a path P from u to v contains a vertex u' from a gadget $T_{i'}$ with $i' \ne i$, $i' \ne j$ and $|i'-i| + |j-i'| = |i-j|$ then the induction hypothesis implies that the length of P is at least $|i'-i|r + |j-i'|r = |i-j|r$. If P contains no such vertices as intermediate vertices then P must contain at least one edge with one endpoint in $T_{i'}$ and another in $T_{j'}$ such that $|i'-j'| \ge |i-j|$. In this case the length of P is at least $|i'-j'|r \ge |i-j|r$, concluding the proof that the distance between a vertex u in a gadget T_i and a vertex v in a gadget T_j is at least $r|i-j|$.

It remains to show that $D_{G'}(c_t c_1') > L$. If a shortest path P from c_t to c_1' avoids the edge $c_t c_1'$, the first vertex in P after c_t must be a vertex u in T_1, and the last vertex in P before c_1' must be a vertex v in T_q. Thus, by the discussion in the previous paragraph, the length of P is at least $2g + (q-1)r \geq qr = 10qb = L$, concluding the proof. □

Lemma 3.2. *If G is 3-colorable then there is an embedding f of $M(G')$ into the line with distorion at most d.*

Proof. Let g be a proper 3-coloring of the vertices of G. We extend g to also color the edges, such that every edge gets a color different from its two endpoints. We give an ordering of the vertices of G', and the embedding f of G' into the line is the pushing embedding imposed by this ordering. We order the vertices of G' as follows, $C_1, T_1, s_1, T_2, s_2, \cdots T_q, C_2$ where the vertices inside C_1 and C_2 are ordered like $\{c_1, \cdots c_t\}$ and $\{c_1', \cdots c_t'\}$ respectively and the vertices inside each gadget T_i are ordered by color with the vertex corresponding to the vertex (or edge) of G colored with 1 comes first.

Observation 2.2 implies that the embedding f is non-contracting. Thus, it suffices to show that the expansion of f is at most d. Because of Lemma 3.1 it suffices to show that $|f(u) - f(v)| \leq w(uv)d$ for every edge $uv \in E'$. For edges with both endpoints in C_1 or both endpoints in C_2 this inequality holds. For an edge uv between a separator vertex and a vertex in a gadget T_i we have $|f(u) - f(v)| \leq g + 2 \leq dg = dw(uv)$. For an edge uv between two vertices in the same gadget T_i we have $|f(u) - f(v)| \leq 2 \leq dw(uv)$. Now, for the edge $c_t c_1'$, $|f(c_1') - f(c_t)| = (2g+2)q = 10aq = 10qba/b = Ld = w(c_t c_1')d$. Similarly, any edge uv with one endpoint in T_i and the other in T_j for $i \neq j$ has the property that u and v correspond to the same vertex (or edge) of G and thus are given the same color by g. Hence $|f(c_1') - f(c_t)| = (2g+2)|i-j| = 10a|i-j| = 10b|i-j|a/b = r|i-j|d = w(uv)d$. As all edges of G' now are accounted for this means that the expansion of f is at most d. □

Lemma 3.3. *If there is an embedding f of $M(G')$ into the line with distorion at most d then G is 3-colorable.*

Proof. Without loss of generality, f is a pushing embedding. Let σ be the ordering of the vertices of G that imposes f. We first prove that σ orders the vertices such that the vertices of the clique C_1 appear consequtively. By consequtively we here mean that no vertices that are not in C_1 come in between the vertices of C_1 in σ. Let u be the leftmost vertex of C_1 as ordered by σ and v be the rightmost vertex of C_1. We know that $|f(u) - f(v)| \geq |C_1| - 1$. However $c_1 c_t$ is the only edge in C_1 satisfying $|f(u) - f(v)| \leq w(uv)d$. Furthermore $w(c_1 c_t)d = |C_1| - 1$ so σ must order the vertices of C_1 consequtively with c_1 and c_t as its endpoints. Similarly σ must order the vertices of C_2 consequtively with c_1' and c_t' as its endpoints. Also, without loss of generality we can assume that C_1 appears before C_2 in our ordering, because if C_2 appears first we can reverse our ordering to obtain another embedding with distortion at most d where C_1 comes first. Now, if c_t is the leftmost vertex of C_1 or c_1' is the rightmost vertex of C_2 the edge $c_t c_1'$ is stretched by a factor more than d. Thus c_t is the rightmost endpoint of C_1 and c_1' is the leftmost endpoint of C_2. Now, every vertex not in C_1 or C_2 has to appear in between C_1 and C_2 because no edge with at least one endpoint outside of $C_1 \cup C_2$ is long enough to strech over the entire expance of C_1 or C_2.

We now prove that σ orders the vertices as follows $C_1, T_1, s_1, T_2, s_2, T_3 \cdots T_q, C_2$. To do this, we need to introduce gaps. A *gap* between two vertcices u and v that appear consequtively in σ is simply the interval $[f(u), f(v)]$ on the real line. We say that a gap is incident to a vertex u if the vertex u is one of the endpoints of the gap. The size of the gap is $|f(u)-f(v)|$. In the layout, there are $4q-1$ vertices and $4q$ gaps that appear between c_t and c_1'. In the following discussion we will treat c_t and c_1' as separator vertices. Each gap that is incident to two separator vertices has size at least $2g$, each gap incident to at least one separator vertex has size at least g and all other gaps have size at least 1. Let x_0, x_1 and x_2 be the number of gaps incident to 0, 1 and 2 separator vertices respectively. Then $|f(c_t)-f(c_1')| \geq 2gx_2+gx_1+x_0$ and $x_0 = 4q-x_2-x_1$. Furthermore each separator vertex (except c_t and c_1') is incident to exactly two gaps, while c_t and c_1' are incident to exactly one gap each among the gaps between c_t and c_1'. Therefore we have that $x_1 + 2x_2 = 2q$. From this it follows that $x_0 = 2q+x_2$ and that $|f(c_t)-f(c_1')| \geq 2gq+x_0$. Hence, if $x_2 > 0$ we have $|f(c_t)-f(c_1')| > 2gq+2q = 2(5a-1)q+2q = 10aq = 10aqb/b = 10qbd = 10Ld = w(c_tc_1')d$ contradicting that the expansion of f is at most d. Thus $x_2 = 0$, $x_1 = x_0 = 2q$ and so $|f(c_t) - f(c_1')| \geq 2gq + 2q = w(c_tc_1')d$. Also, if any gap not incident to any separator vertices has size more than 1, or if any of the gaps incident to a separator vertex have size more than g then $|f(c_t) - f(c_1')| > 2gq + 2q = w(c_tc_1')d$, again contradicting that the expansion of f is at most d. Finally one should note that $g > d$ so no edge with weight one can ever be stretched over a gap of size g. Since the only edges of weight 1 in G' are within a gadget T_i and every edge incident to a separator vertex has weight g this implies that σ must order the vertices in the aforementioned order $C_1, T_1, s_1, T_2, s_2, T_3 \cdots T_q, C_2$.

For a vertex v in V, if there is a vertex v' in a gadget T_i corresponding to v, we look at the position that v' is assigned by σ compared to the other vertices of T_i. If v' is the leftmost vertex of T_i we say that the *color* of v in the gadget T_i is 1. Similarly, if v' is the second or third vertex of T_i we say that the color of v in T_i is 2 or 3 respectively. In all of these cases we say that v has a color in T_j. We prove that for any i,j with $i < j$ and vertex $v \in V$ such that v has a color in T_i and in T_j the color of v in T_j is smaller or equal to the color of v in T_i. Suppose this is not the case, and let u' and v' be the vertices corresponding to v in gadgets T_i and T_j such that the color of v in T_j is strictly greater than the color of u in T_i. Then we know that $|f(v') - f(u')| > (2g + 2)|j - i| = (2(5a - 1) + 2)|j - i| = 10a|j - i|b/b = 10b|j - i|d = r|j - i|d$. However since both u' and v' correspond to v there is an edge of weight $r|j - i|$ between u and v that is stretched more than d by the embedding. This is a contradiction and we conclude that the color of v in T_j must be less than or equal to the color of v in T_i. Notice that since a vertex (in a gadget) can have one of three different colors this implies that as we scan the gadgets from T_1 to T_q the color of a vertex can change at most twice. Thus, there must be some $0 \leq p < 2n + 1$ such that every vertex of G has the same color in all gadgets it appears in among the gadgets from T_{1+mp} to $T_{m(p+1)}$. Notice that every vertex and every edge of G has a color in at least one of these gadgets. We can now make a coloring g of the vertices of G such that for every vertex $v \in V$ and every i between $1 + mp$ and $m(p+1)$ such that v has a color in T_i, the color if v in T_i is exactly $g(v)$. All that remains to prove is that g is a proper coloring. This is easy, for every edge $uv \in E$ there is an i between $1 + mp$ and $m(p + 1)$ such that the edge uv has a color in T_i. Then both u and v have colors in T_i and their colors in T_i must be different. Since $g(u)$ is equal to us color in T_i and $g(v)$ is equal to vs color in T_i this implies $g(u) \neq g(v)$ concluding the proof. □

Together with the construction of G' from G, Lemmas 3.2 end 3.3 immediately imply the following theorem.

Theorem 3.4. *Embedding general metrics into the line with distorion at most d is NP-*

Complete for every fixed rational $d \geq 2$.

4 Embedding Graphs into Trees of Bounded Degree

Given a graph G with shortest path metric D_G and a tree T with maximum degree Δ, having shortest path metric D_T, we give an algorithm that decides whether G can be embedded into T with distortion at most d in time $n^2 \cdot |V(T)| \cdot 2^{O((5d)^{\Delta^{d+1}} \cdot d)}$. We assume that the tree T is rooted, and we will refer to the root of T as $r(T)$. For a vertex v in the tree, T_v is the subtree of T rooted at v, and $C(v)$ is the set of v's children. Finally, for an edge uv of T, let $T_u(uv)$ and $T_v(uv)$ be the tree of $T \setminus uv$ that contains u and v respectively. Notice that if u is the parent of v in the tree, then $T_v(uv) = T_v$ and $T_u(uv) = T \setminus V(T_v)$. As in the previous section, we need to define feasible partial embeddings together with the notion of succession. For a vertex $u \in V(T)$ and a subset S of $V(G)$, a *u-partial embedding* is a function $f_u : S \to B(u, d+1)$.

Definition 4.1. *For a u-partial embedding f_u of a subset $S \subseteq V(G)$ and a vertex $v \in N(u)$ we define $S[v, f_u] = \{x \in S \ : \ f_u(x) \in V(T_v(uv))\}$. Given two integers i and j, $0 \leq i \leq j \leq k$, let $S^{[i,j]}[f_u] = \{x \in S \ : \ i \leq D(f_u(x), u) \leq j\}$. Finally, let $S^{[i,j]}[v, f_u] = S^{[i,j]}[f_u] \cap S[v, f_u]$, $S^k[v, f_u] = S^{[k,k]}[v, f_u]$ for $k \geq 1$ and $S^0[f_u] = S^{[0,0]}[f_u]$.*

Definition 4.2. *For a u-partial embedding f_u of a subset $S \subseteq V(G)$ and a vertex $v \in N(u)$ we define $M[v, f_u]$ to be the union of the vertex sets of all connected components of $G \setminus S$ that have neighbors in $S[v, f_u]$.*

Definition 4.3. *A u-partial embedding f_u of a subset S of $V(G)$ is called feasible if*

1. *f_u is a non-contracting distortion d embedding of S into $B(u, d+1)$;*
2. *For any distinct pair $v, w \in N(u)$, $M[v, f_u] \cap M[w, f_u] = \emptyset$;*
3. *$N(S^0[f_u]) \subseteq S$.*

Definition 4.4. *For a feasible u-partial embedding f_u of a subset S_u of $V(G)$ and a feasible v-partial embedding f_v of a subset S_v of $V(G)$ with $v \in C(u)$ we say that f_v* succeeds *f_u if*

1. *$S_u \cap S_v = (S_u^{[0,d]}[f_u] \cup S_u^{d+1}[v, f_u]) = (S_v^{[0,d]}[f_v] \cup S_v^{d+1}[u, f_v])$;*
2. *for every $x \in S_u \cap S_v$, $f_u(x) = f_v(x)$;*
3. *$M[v, f_u] = \bigcup_{x \in N(v) \setminus u} (M[x, f_v] \uplus S_u^{d+1}[x, f_v])$;*
4. *$M[u, f_v] = \bigcup_{x \in N(u) \setminus v} (M[x, f_u] \uplus S_v^{d+1}[x, f_u])$.*

Suppose we have picked out a subtree T_v for a vertex $v \in V(T)$ and found a non-contracting embedding f' with distortion at most d of a subset Z of $V(G)$ into $T' = T[\bigcup_{u \in V(T_v)} B(u, d+1)]$. We want to complete the embedding of G into T starting from this point. That is, we wish to find a non-contracting distortion d embedding of G into T such that for every vertex u with $f(u) \in V(T')$, we have that $u \in Z$ and such that if $u \in Z$ then $f(u) = f'(u)$. At this point, a natural question arises. Can we impose constraints on the restriction of f to $V(T) \setminus V(T_v)$ such that f restricted to $V(T) \setminus V(T_v)$ satisfies these conditions if and only if f is a non-contracting distortion d embedding of G into T? One necessary condition is that f restricted to $V(T) \setminus V(T_v)$ must be a non-contracting distortion d embedding of $\{u \in V(G) \ : \ f(u) \in V(T) \setminus V(T_v)\}$. We can obtain another condition by applying the definition of feasible u-partial embeddings. For each vertex

u, we can use arguments similar to the ones in Section 2 in order to determine which connected components of $T \setminus V(T_v)$ f must map u to in order to be a non-contracting distortion d embedding of G into T.

As we have seen in Section 2, for the line, these two conditions are both necessary and sufficient. Unfortunately, for the case of bounded degree trees, this is not the case. The reason the conditions are sufficient when we restrict ourselves to the line is that every embedding of a graph metric into the line that is *locally* non-contracting and *locally* expanding by a factor at most d, also is *globally* non-contracting and expanding by a factor at most d. When we embed into trees of bounded degree, every embedding that is locally expanding by a factor at most d, also has this property globally. However, every locally non-contracting embedding needs not be *globally* non-contracting. To cope with this issue of non-contraction, we introduce the concept of vertex types. Intuitively, vertices of the same type in T_v are indistinguishable when viewed from $T \setminus V(T_v)$. We show that the set of possible vertex types can be bounded by a function of d and Δ. Then, to complete f from f' we only need to know the restriction of f' to $B(v, d+1)$ and which vertex types appear in T_v. This will imply that the amount of information we need to pass on from f' to f is bounded by $n \cdot h(d, \Delta)$. We exploit this fact to give a dynamic programming algorithm for the problem. In the rest of this section, we formalize this intuition.

For a vertex $u \in T$, a neighbour v of u and a feasible u-partial embedding f_u of a subset S of $V(G)$ we define a $[v, f_u]$-*type* to be a function $t : S[v, f_u] \to \{\infty, 3d + 2, d, \cdots, -d, -(d+1)\}$ and a $[v, f_u]$-*typelist* to be a set of $[v, f_u]$-*types*. For an integer k let $\beta(k) = k$ if $k \leq 3d + 2$ and $\beta(k) = \infty$ otherwise.

Definition 4.5. *For a vertex $u \in T$ with two neighbours v and w, and a feasible u-partial embedding f_u of a subset S of $V(G)$ together with a $[v, f_u]$-typelist $\mathcal{L}_1$ and a $[w, f_u]$-typelist $\mathcal{L}_2$ we say that $\mathcal{L}_1$ and $\mathcal{L}_2$* agree *if for every type $t_1 \in \mathcal{L}_1$ and $t_2 \in \mathcal{L}_2$ there is a vertex $x \in S[v, f_u]$ and a vertex $y \in S[w, f_u]$ such that $t_1(x) + t_2(y) \geq D_G(x, y)$.*

Definition 4.6. *For a vertex $u \in T$, a neighbour v of u, a feasible u-partial embedding f_u of a subset S of $V(G)$ and a $[v, f_u]$-typelist $\mathcal{L}$ we say that $\mathcal{L}$ is* compatible *with $S[v, f_u]$ if for every vertex x in $S[v, f_u]$ there is a type $t \in \mathcal{L}$ such that for every $y \in S[v, f_u]$, $D_T(f_u(x), u) - D_G(x, y) = t(y)$.*

Definition 4.7. *A* feasible u-state *is a feasible partial embedding f_u of a subset S of $V(G)$ together with a $[v, f_u]$-typelist $\mathcal{L}[v, f_u]$ for every $v \in N(u)$ such that the following conditions are satisfied:*

1. *$\mathcal{L}[v, f_u]$ is compatible with $S[v, f_u]$ for every $v \in N(u)$;*
2. *For every pair of distinct vertices x and y in $N(u)$, $\mathcal{L}[x, f_u]$ agrees with $\mathcal{L}[y, f_u]$.*

Definition 4.8. *Let $u \in V(T)$, $v \in C(u)$. Let $\mathcal{X}_u$ be a feasible u-state and $\mathcal{X}_v$ be a feasible v-state. We say that $\mathcal{X}_v$* succeeds *$\mathcal{X}_u$ if*

1. *f_v succeeds f_u;*
2. *For every $w \in (N(v) \setminus u)$ and a type $t_1 \in \mathcal{L}[w, f_v]$ there is a type $t_2 \in \mathcal{L}[v, f_u]$ such that*
   - *(a) For every node $x \in S[v, f_u] \cap S[w, f_v]$, $t_2(x) = \beta(t_1(x) + 1)$;*
   - *(b) For every node $x \in (S[v, f_u] \setminus S[w, f_v])$, $t_2(x) = \beta(\max_{y \in S[w, f_v]}(t_1(y) + 1 - D_G(x, y)))$.*

3. *For every $w \in (N(u) \setminus v)$ and a type $t_1 \in \mathcal{L}[w, f_u]$ there is a type $t_2 \in \mathcal{L}[u, f_v]$ such that*

   (a) *For every node $x \in S[u, f_v] \cap S[w, f_u]$, $t_2(x) = \beta(t_1(x) + 1)$;*

   (b) *For every node $x \in (S[u, f_v] \setminus S[w, f_u])$, $t_2(x) = \beta(\max_{y \in S[w, f_u]}(t_1(y) + 1 - D_G(x, y)))$.*

Lemma 4.9. *If there is a distortion d embedding F of G into T then, for every vertex u of $V(T)$ there is a feasible u-state $\mathcal{X}_u$ such that for every vertex $v \in V(T), w \in C(v)$, $\mathcal{X}_w$ succeeds $\mathcal{X}_v$.*

Proof. We start by giving a feasible u-partial embedding f_u for each vertex of the tree. Recall that a feasible u-state contains a feasible u-partial embedding f_u of a subset S_u of $V(G)$. For a vertex $u \in V(T)$ we define f_u to be the restriction of F to $B(u, d+1)$. It is easy to see that f_u indeed is a feasible partial embedding for every u and that for every vertex $v \in V(T), w \in C(v)$, f_w succeeds f_v.

Now, for every vertex $u \in V(T)$ and $v \in N(v)$ we give a typelist $\mathcal{L}[v, f_u]$. For every vertex $x \in (S[v, f_u] \cup M[v, f_u])$ we add a $[v, f_u]$-type $t_x[v, f_u]$ to $\mathcal{L}[v, f_u]$. For a vertex $y \in S[v, f_u]$, $t_x[v, f_u](y) = \beta(D_T(F(x), u) - D_G(x, y))$. Notice that since $y \in B(u, d+1)$ and F is non-contracting, by the triangle inequality it follows that $t_x(y) \geq -(d+1)$ and thus t_x is a $[v, f_u]$-type. Furthermore, for every $u \in V(T)$, $\mathcal{L}[v, f_u]$ is compatible with $S[v, f_u]$ because for every x and y in $S[v, f_u]$ we have that $t_x[v, f_u](y) = \beta(D_T(F(x), u) - D_G(x, y))$. In order to show that each state $\mathcal{X}_u$ is a feasible u-state it remains to show that for every vertex $u \in V(T)$ and every pair of distinct vertices v and w in $N(u)$, $\mathcal{L}[v, f_u]$ agrees with $\mathcal{L}[w, f_u]$. Assume for contradiction that there is a type $t_a[v, f_u] \in \mathcal{L}[v, f_u]$ and a type $t_b[w, f_u] \in \mathcal{L}[w, f_u]$ such that $t_a[v, f_u](x) + t_b[w, f_u](y) < D_G(x, y)$ for every $x \in S[v, f_u]$ and $y \in S[w, f_u]$. Let $x' \in S[v, f_u]$ and $y' \in S[w, f_u]$ be the pair of vertices that maximizes $t_a[v, f_u](x') + t_b[w, f_u](y') - D_G(x', y')$. There is a vertex $a \in (S[v, f_u] \cup M[v, f_u])$ and a vertex $b \in (S[w, f_u] \cup M[w, f_u])$ such that $\beta(D_T(f(a), u) - D_G(a, x)) = t_a[v, f_u](x)$ for every $x \in S[v, f_u]$ and $\beta(D_T(F(b), u) - D_G(b, y)) = t_b[w, f_u](y)$ for every $y \in S[w, f_u]$. This yields $D_T(F(a), u) - D_G(a, x') + D_T(F(b), u) - D_G(b, y') = (D_T(F(a), u) + D_T(F(b), u)) - (D_G(a, x') + D_G(b, y')) < D_G(x', y')$. Now, $(D_T(F(a), u) + D_T(F(b), u)) = D_T(F(a), F(b))$ since u lies on the unique $f(a)$-$f(b)$ path in T. Also, Since x' and y' are the pair that maximize $t_a[v, f_u](x') + t_b[w, f_u](y') - D_G(x', y')$ and every shortest x'-y' path in G must pass both through $S[v, f_u]$ and $S[w, f_u]$, we conclude that $(D_G(a, x') + D_G(b, y') + D_G(x', y')) = D_G(a, b)$. However this implies $D_T(F(a), F(b)) < D_G(a, b)$ contradicting that F is non-contracting.

It remains to prove that for every vertex $u \in T$, $v \in N(u)$, $w \in (N(v) \setminus u)$ and type $t_1 \in \mathcal{L}[w, f_v]$ there is a type $t_2 \in \mathcal{L}[v, f_u]$ such that

1. for every node x in $S[v, f_u] \cap S[w, f_v]$, $t_2(x) = \beta(t_1(x) + 1)$;

2. for every node x in $S[v, f_u] \setminus S[w, f_v]$, $t_2(x) = \beta(\max_{y \in S[w, f_v]}(t_1(y) + 1 - D_G(x, y)))$.

Let $t_a[w, f_v] \in \mathcal{L}[w, f_v]$, and let a be the vertex of $S[w, f_v] \cup M[w, f_v]$ such that for every x in $S[w, f_v]$, $t_a[w, f_v](x) = \beta(D_T(F(a), v) - D_G(a, x))$. Now, $S[w, f_v] \cup M[w, f_v] \subseteq S[v, f_u] \cup M[v, f_u]$ so $a \in (S[v, f_u] \cup M[v, f_u])$. Let $t'_a[v, f_u]$ be the type in $\mathcal{L}[v, f_u]$ so that for every x' in $S[v, f_u]$, $t'_a[v, f_u](x') = \beta(D_T(F(a), u) - D_G(a, x'))$. As $\beta(D_T(F(a), u)) = \beta(D_T(F(a), v) + 1)$ it is easy to see that for every node $x \in (S[v, f_u] \cap S[w, f_v])$, $t'_a[v, f_u](x) = \beta(t_a[w, f_v](x) + 1)$. Finally, observe that for a vertex $x \in (S[v, f_u] \setminus S[w, f_v])$ every a-x path in G must pass through $S[w, f_v]$. Thus $D_G(a, x) = \min_{y \in S[w,f_v]} D_G(a, y) + D_G(x, y)$ and so $t'_a[v, f_u](x) = \beta(\max_{y \in S[w,f_v]}(t_a[w, f_v](y) + 1 - D_G(x, y)))$. This concludes the proof. □

The main result of this section relies heavily on the following lemma. The proof of this lemma has been moved to the appendix, due to space restrictions.

Lemma 4.10. *If there is a feasible u-state $\mathcal{X}_u$ for every vertex u of $V(T)$ such that for every vertex $v \in V(T), w \in C(v)$, $\mathcal{X}_w$ succeeds $\mathcal{X}_v$ then there is a distortion d embedding F of G into T.*

Theorem 4.11. *There is an algorithm that given a graph G, tree T with maximum degree Δ and an integer d decides whether G can be embedded into T with distortion at most d in time $n^2 \cdot |V(T)| \cdot 2^{O((5d)^{\Delta^{d+1}} \cdot d)}$.*

Proof. The algorithm proceeds as follows. First, check that $\Delta(G) \le \Delta^d$ (follows from local density argument). Now, we do bottom up dynamic programming on the tree T. For each vertex u of the tree we make a boolean table with an entry for each possible feasible u-state. For every leaf of the tree all the entries are set to true. For an inner node u and a feasible u-state $\mathcal{X}_u$ we set $\mathcal{X}_u$'s entry to true if for each child v of u there is a feasible v-state $\mathcal{X}_v$ that succeeds $\mathcal{X}_u$ and so that $\mathcal{X}_v$'s entry is set to true. The algorithm returns "yes" if, at the termination of this procedure, there is a feasible $r(T)$-state $\mathcal{X}_{r(T)}$ with its table entry set to true. The algorithm clearly terminates, and correctness of this algorithm follows from Lemmas 4.10 and 4.9.

We now proceed to the running time analysis. In our bottom up sweep of T, we consider every edge and every vertex of T exactly once, which yields a factor of $n_t = |V(T)|$. For each vertex u we consider each feasible u-state $\mathcal{X}_u$ once, and for each such state and every child v of u of the state we need to enumerate all feasible v-states that succeed $\mathcal{X}_u$. In fact, we enumerate a larger set of candidate feasible v-states and for each such state $\mathcal{X}_v$ we check whether $\mathcal{X}_v$ succeeds $\mathcal{X}_u$.

First we show that the number of feasible u-partial embeddings is at most $n \cdot \Delta^{O(d^2 \cdot \Delta^{d+1})}$. This follows from the fact that for any vertex u of the tree $|B(u, d+1)| \le \Delta^{d+1}$ and that the domain of a any feasible u-partial embedding f_u is contained in a ball of radius at most $2d + 2$ in G. Because the degree of G is bounded, a ball of radius $2d + 2$ in G can contain at most $\Delta^{O(d^2)}$ vertices.

One can easily prove that if the feasible partial embedding f_u is given, the number of types and typelists that can appear in a feasible u-state together with f_u is bounded by $(5d)^{\Delta^{d+1}}$ and $2^{O((5d)^{\Delta^{d+1}})}$ respectively. Thus, the number of feasible u-states is bounded by $2^{O((5d)^{\Delta^{d+1}} \cdot d)}$. If the domain S_v of a feasible partial embedding f_v for a child v of u is non-empty then we can use the fact that S_v must have a non-empty intersection with the domain of f_u to bound the number of potential successors of a u-state by $2^{O((5d)^{\Delta^{d+1}} \cdot d)} \cdot \Delta^d \le 2^{O((5d)^{\Delta^{d+1}} \cdot d)}$. Since we can check whether a particular u-feasible state succeeds another in time $n \cdot 2^{O((5d)^{\Delta^{d+1}} \cdot d)}$ the overall running time of the algorithm is bounded by $n^2 n_t \cdot 2^{O((5d)^{\Delta^{d+1}} \cdot d)}$. □

5 Concluding Remarks and Open Problems

In this paper we describe a result that might be considered surprising: embedding a general unweighted graph minimum distance metric into a tree metric for a tree of maximum degree

Δ, parameterized by (Δ, d) where d is the distortion is *fixed-parameter tractable*, a strong improvement on the best previous results, and a result that invites further investigation of this subject from the parameterized point of view. There are two questions that arise immediately:

- What is the parameterized complexity of the problem if we parameterize only by d?
- What is the parameterized complexity of embedding graphs into target metrics that are minimum distance metrics of graphs with maximum degree Δ and treewidth at most w, parameterizing by (Δ, d, w)?

We believe that both of these problems are $W[1]$-hard. In fact, we suspect that these problems are NP-hard, even for some fixed values of d, δ and w.

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6 Appendix

Proof of Lemma 4.10: *If there is a feasible u-state $\mathcal{X}_u$ for every vertex u of $V(T)$ such that for every vertex $v \in V(T), w \in C(v)$, $\mathcal{X}_w$ succeeds $\mathcal{X}_v$ then there is a distortion d embedding F of G into T.*

Proof. For every vertex u, let f_u be the feasible u-partial embedding of the subset $S_u \subseteq V(G)$ in $\mathcal{X}_u$. We prove the lemma by proving a series of claims

Claim 6.1. *For every vertex $x \in V(G)$ there is a $u \in V(T)$ such that $x \in S_u$.*

Proof. If $x \in S_{r(T)}$ we are done, so assume that $x \notin S_{r(T)}$. This means that $x \in \bigcup_{v\in C(r(T))} M[v, f_{r(T)}]$. Let $v_1 \in C(r(T))$ be the vertex so that $x \in M[v_1, f_{r(T)}]$. Observe that the choice of v_1 implies that $x \notin M[r(T), f_{v_1}]$. Now, if $x \in S_{v_1}$ we are done, otherwise $x \in \bigcup_{v\in C(v_1)} M[v, f_{v_1}]$. Let v_2 be the vertex in $C(v_1)$ so that $x \in M[v_2, f_{v_1}]$. Again, the choice of v_1 implies that $x \notin M[v_1, f_{v_2}]$. If $x \in S_{v_2}$ we are done, otherwise we can select v_3, v_4 and so on until we select a leaf v_q. The choice of v_q implies that $x \in S_{v_q} \cup \bigcup_{v\in C(v_q)} M[v, f_{v_q}] = S_{v_q}$. □

Claim 6.2. *For every vertex $x \in V(G)$, the set $\{u \in V(T) : x \in S_u\}$ induces a connected subtree of T.*

Proof. Suppose for contradiction that this is not the case. Then there is a pair of vertices $u, v \in V(T)$ such that $x \in S_u$, $x \in S_v$, $uv \notin E(T)$ and for every w on the u-v-path in T, $x \notin S_w$. Let w' and w'' be the predecessor and successor of w on the u-v path respectively. By the properties of succession of feasible u-partial embeddings both $M[w', f_w]$ and $M[w'', f_w]$ must contain x. This contradicts that f_w is a feasible partial embedding. □

¿From Claim 6.2 together with property 2 of succession feasible u-partial embeddings it is clear that for every pair of vertices u and v in $V(T)$ such that $x \in S_u$ and $x \in S_v$, $f_u(x) = f_v(x)$. We can therefore define a function $F : V(G) \to V(T)$ such that for every $x \in V(G)$ and $u \in V(T)$ it holds that if $x \in S_u$ then $F(x) = f_u(x)$. This property also guarantees F maps distinct vertices of G onto distinct vertices of T. In the rest of the proof of Lemma 4.10 we will prove that the expansion of F is at most d and that F is non-contracting.

Claim 6.3. *The expansion of F is at most d.*

Proof. It suffices to prove that F expands every edge of G by at most a factor of d. Let $xy \in E(G)$. Let $u = F(x)$. By the property 3 of feasible u-partial embeddings $y \in S_u$. Furthermore, since f_u is a feasible u-partial embedding, $D_T(F(x), F(y)) = D_T(f_u(x), f_u(y)) \leq d$ which completes the proof □

We now proceed to prove that F is non-contracting.

Claim 6.4. *For every path $P = v_1v_2...v_k$ in T, with $v_1 = u$ and $v_k = w$ the following must apply.*

1. *F restricted to $\bigcup_{v_i\in P} S_{v_i}$ is non-contracting.*
2. *For every vertex $x \in S_w$ one of the following two conditions must hold.*
   (a) *either there is a $v_j \in P$, $y \in S_{v_j}$ such that $D_T(F(x), v_j) - D_G(x, y) > 3d + 2$*

(b) *or there is a type* $t_x[v_2, f_u] \in \mathcal{L}[v_2, f_u]$ *such that for every* $y \in S[v, f_u]$, $t_x[v_2, f_u](y) = D_T(F(x), u) - D_G(x, y)$.

Proof. We prove the claim by induction on k. If $k = 1$ then (1) is true because f_u is a feasible partial embedding and (2b) holds because of the compatibility constraints of feasible u-states.

For $k = 2$, we first prove that (2b) holds for every $x \in S_w$. If $x \in S_u$ then (2b) holds because of the compatibility constraints of feasible u-states. Therefore, consider a vertex $x \in S_w \setminus S_u$. Then $x \in S_w^{d+1}[w', f_w]$ for a $w' \in (N(w) \setminus u)$. By compatibility, there is a type $t_1[w', f_w] \in \mathcal{L}[w', f_w]$ such that for every y in S_w, $t_1[w', f_w](y) = D_T(F(x), w) - D_G(x, y)$. By the properties of succession of feasible u-states, there is a type $t_2[w, f_u] \in \mathcal{L}[w, f_u]$ such that

1. For every node y in $S[w, f_u] \cap S[w', f_w]$,

$$\begin{aligned} t_2[w, f_u](y) &= \beta(t_1[w', f_w](y) + 1) \\ &= t_1[w', f_w](y) + 1 \\ &= D_T(F(x), w) - D_G(x, y) + 1 \\ &= D_T(F(x), u) - D_G(x, y). \end{aligned}$$

2. For every node y in $S[w, f_u] \setminus S[w', f_w]$,

$$\begin{aligned} t_2[w, f_u](y) &= \beta\left(\max_{z \in S[w', f_w]} (t_1[w', f_w](z) + 1 - D_G(z, y))\right) \\ &= \max_{z \in S[w', f_w]} \big(t_1[w', f_w](z) + 1 - D_G(z, y)\big) \\ &= \max_{z \in S[w', f_w]} (D_T(F(x), w) - D_G(x, z) + 1 - D_G(z, y)) \\ &= D_T(F(x), u) - D_G(x, y). \end{aligned}$$

Thus (2b) holds for every x in S_w. Using this fact we can now prove (1). Observe that it is sufficient to prove that F does not contract any vertex $y \in (S_u \setminus S_w)$ and $x \in (S_w \setminus S_u)$. Let u' be the neighbour of u such that $y \in S[u', f_u]$. By (2b) there is a type $t_x[w, f_u]$ such that for every $z \in S[w, f_u]$, $t_x[w, f_u](z) = D_T(F(x), u) - D_G(x, z)$. By the properties of feasible u-states there is a type $t_y[u', f_u]$ such that for every $z \in S[u', f_u]$, $t_y[u', f_u](z) = D_T(F(y), u) - D_G(y, z)$. Since $t_x[w, f_u]$ and $t_y[u', f_u]$ must agree, it follows that there is a vertex $x' \in S[w, f_u]$, and a vertex $y' \in S[u', f_u]$ such that $t_x[w, f_u](x') + t_y[u', f_u](y') \geq D_G(x', y')$. By substituting for $t_x[w, f_u](x')$ and $t_y[u', f_u](y')$ we obtain

$$\begin{aligned} D_T(F(x), F(y) - D_G(x, y) &\geq \big(D_T(F(x), u) - D_G(x, x')\big) \\ &\quad + \big(D_T(F(y), u) - D_G(y, y')\big) - D_G(f(x'), f(y') \\ &\geq 0. \end{aligned}$$

Finally, suppose the statement of the claim holds for every $k' < k$ for some $k > 2$. We prove that the statement also must hold for k. We start by showing (2). Consider a vertex $x \in S_w$ such that for every v_j, $j \geq 1$ and every $y \in S_{v_j}$ we have that $D_T(F(x), v_j) - D_G(x, y) \leq 3d + 2$, that is, (2$a$) does not hold for x. We need to show that (2b) must hold for x. By the inductive hypothesis there is a type $t_x[v_3, f_{v_2}] \in \mathcal{L}[v_3, f_{v_2}]$ so that for every

$y \in S[v_3, f_{v_2}]$, $t_x[v_3, f_{v_2}](y) = D_T(F(x), v_2) - D_G(x, y)$. Furthermore, by the assumption that $(2a)$ does not hold for x, $t_x[v_3, f_{v_2}] \leq 3d + 2$. By the properties of succession of feasible u-states there must be a type $t'_x[v_2, f_u] \in \mathcal{L}[v_2, f_u]$ such that:

1. For every node y in $(S[v_2, f_u] \cap S[v_3, f_{v_2}])$:

$$\begin{aligned} t'_x[v_2, f_u](y) &= \beta(t_x[v_3, f_{v_2}](y) + 1) \\ &= \beta\left(D_T(F(x), v_2) - D_G(x, y) + 1\right) \\ &= \beta\left(D_T(F(x), u) - D_G(x, y)\right). \end{aligned}$$

   Observe that if $\beta(D_T(F(x), u) - D_G(x, y)) = \infty$ then $D_T(F(x), u) - D_G(x, y) > 3d + 2$ which implies that $(2a)$ holds for x which is a contradiction. Therefore, $\beta(D_T(F(x), u) - D_G(x, y)) \neq \infty$ so $\beta(D_T(F(x), u) - D_G(x, y)) = D_T(F(x), u) - D_G(x, y)$.

2. For every node y in $S[v_2, f_u] \setminus S[v_3, f_{v_2}]$,

$$t'_x[v_2, f_u](y) = \beta\left(\max_{z \in S[v_3, f_{v_2}]} (t_x[v_3, f_{v_2}](z) + 1 - D_G(z, y))\right).$$

   As before, $t'_x[v_2, f_u](y) \leq 3d + 2$ because otherwise $(2a)$ holds for x. Thus

$$\begin{aligned} t'_x[v_2, f_u](y) &= \beta\left(\max_{z \in S[v_3, f_{v_2}]} (t_x[v_3, f_{v_2}](z) + 1 - D_G(z, y))\right) \\ &= \max_{z \in S[v_3, f_{v_2}]} \left(t_x[v_3, f_{v_2}](z) + 1 - D_G(z, y)\right). \end{aligned}$$

   Following this

$$\begin{aligned} t'_x[v_2, f_u](y) &= \max_{z \in S[v_3, f_{v_2}]} \left(D_T(F(x), v_2) - D_G(x, z) + 1 - D_G(z, y)\right) \\ &= D_T(F(x), u) - D_G(x, y). \end{aligned}$$

Thus $(2b)$ holds for x and (2) is true for $|P| = k$. It remains to prove that (1) is true for $|P| = k$ as well. It is sufficient to prove that F does not contract any $x \in (S_u \setminus S_{v_2})$ with any $y \in (S_w \setminus S_{v_{k-1}})$. There are two cases, either $(2a)$ holds for y or $(2a)$ does not, and in that case, $(2b)$ holds for y. In the latter case let $t_y[v_1, f_u]$ be the type in $\mathcal{L}[v_1, f_u]$ such that for every z in $S[v_2, f_u]$, $t_y[v_1, f_u](z) = D_T(F(y), u) - D_G(y, z)$. Also, let $u' \in (N(u) \setminus v_1)$ be the neighbour of u such that $x \in S[u', f_u]$. As in the proof of (1) for $k = 2$, let $t_x[u', f_u]$ be the type in $\mathcal{L}[u', f_u]$ such that for every z in $S[u', f_u]$, $t_x[u', f_u](z) = D_T(F(x), u) - D_G(x, z)$. Again as in the proof of (1) for $k = 2$, $t_x[u', f_u](z)$ and $t_y[v_1, f_u]$ must agree which in turn implies that F does not contract x and y

To conclude, we consider the case when $(2a)$ holds for y. Let v_j be a vertex such that there is a $y' \in S_{v_j}$ so that $D_T(F(y), v_j) - D_G(y, y') > 3d + 2$. By the induction hypothesis, F does not contract x and y'. This gives us the following inequality for $D_T(F(x), F(y))$ and $D_G(x, y)$:

$$\begin{aligned} d_T(F(x), F(y)) &= D_T(F(x), v_j) + D_T(v_j, F(y)) \\ &\geq (D_T(F(x), F(y')) - D_T(v_j, F(y'))) + D_T(v_j, F(y)) \\ &\geq D_G(x, y') + D_G(y, y') + 3d + 2 - D_T(v_j, F(y')) \\ &\geq D_G(x, y) + 2d + 1 \geq D_G(x, y). \end{aligned}$$

This implies that the statement of the claim holds for every positive k. □

The claims together clearly prove the existence of a non-contracting embedding F of G into T with distortion at most d. $\square$