Parameterized Complexity of Stabbing Rectangles and Squares in the Plane

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Abstract. The NP-complete geometric covering problem Rectangle Stabbing is defined as follows: Given a set of horizontal and vertical lines in the plane, a set of rectangles in the plane, and a positive integer k, select at most k of the lines such that every rectangle is intersected by at least one of the selected lines.
While it is known that the problem can be approximated in polynomial time with a factor of two, its parameterized complexity with respect to the parameter k was open so far—only its generalization to three or more dimensions was known to be W[1]-hard. Giving two fixed-parameter reductions, one *from* the W[1]-complete problem Multicolored Clique and one *to* the W[1]-complete problem Short Turing Machine Acceptance, we prove that Rectangle Stabbing is W[1]-complete with respect to the parameter k, which in particular means that there is no hope for fixed-parameter tractability with respect to the parameter k. Our reductions show also the W[1]-completeness of the more general problem Set Cover on instances that "almost have the consecutive-ones property", that is, on instances whose matrix representation has at most two blocks of 1s per row.
For the special case of Rectangle Stabbing where all rectangles are squares of the same size we can also show W[1]-hardness, while the parameterized complexity of the special case where the input consists of rectangles that do not overlap is open. By giving an algorithm running in $(4k+1)^k \cdot n^{O(1)}$ time, we show that Rectangle Stabbing is fixed-parameter tractable in the still NP-hard case where *both* these restrictions apply.

1 Introduction

Geometric covering problems arise in many applications and are intensively studied (see [8, 9, 14]). Here, we consider the problem 2-Dimensional Rectangle Stabbing (Rectangle Stabbing for short), which is defined as follows.

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(2-Dimensional) Rectangle Stabbing

Input: A set L of vertical and horizontal lines embedded in the plane, a set R of axis-parallel rectangles embedded in the plane, and a positive integer k.

Question: Is there a set $L' \subseteq L$ with $|L'| \leq k$ such that every rectangle from R is intersected (*"stabbed"*) by at least one line from L'?

Rectangle Stabbing is NP-complete (see [8]). Its optimization version, considered in the setting of polynomial-time approximation, asks for a *minimum-cardinality* set $L' \subseteq L$ to cover all rectangles from R.

Applications of Rectangle Stabbing range from radiotherapy [10] to embedded sensor networks, spatial data organization and statistical data analysis [1, 11]. Also, the problem of stabbing arbitrary connected figures (instead of rectangles) in the plane with horizontal and vertical lines can easily be reduced to Rectangle Stabbing by replacing each figure by its bounding box. The same holds for the stabbing problem where only the rectangles in the plane are given and a minimum number of horizontal and vertical lines shall be inserted that stab all rectangles: any instance of this problem can be transformed into an instance of Rectangle Stabbing by inserting $O(|R|)$ lines. Without loss of generality, we can always assume that all given lines have integer coordinates.

The literature so far mainly considers the polynomial-time approximability of Rectangle Stabbing and its variants. Hassin and Megiddo [10] give a factor-$d2^{d-1}$ approximation for stabbing d-dimensional, identical objects with axis-parallel lines in the d-dimensional space. Gaur et al. [8] achieve a factor-d approximation for d-Dimensional Rectangle Stabbing, that is, for stabbing d-dimensional, axis-parallel hyperboxes with $(d-1)$-dimensional, axis-parallel hyperplanes; the two-dimensional case Rectangle Stabbing, hence, can be approximated with a factor of two. A similar result was obtained by Mecke et al. [16]; they give a factor-d approximation algorithm for a problem called d-C1P-Set Cover, which is a generalization of d-Dimensional Rectangle Stabbing. Weighted and capacitated versions of d-Dimensional Rectangle Stabbing have been considered by Even et al. [4] and by Xu and Xu [18], also leading to several approximation algorithms. A restricted, but still NP-complete variant of (2-Dimensional) Rectangle Stabbing is called Interval Stabbing; here, every rectangle in the input is intersected by at most one horizontal line (that is, every rectangle is a horizontal interval in the plane). Kovaleva and Spieksma [12, 13] give constant-factor approximations for several variants of Interval Stabbing. Approximation algorithms for the more general variant of Interval Stabbing where the input contains horizontal *and* vertical intervals have been developed by Hassin and Megiddo [10].

Concerning the parameterized complexity of Rectangle Stabbing and d-Dimensional Rectangle Stabbing (that is, the question whether there is an algorithm running in $f(k) \cdot |(L, R, k)|^{O(1)}$ time), it is known that, on the one hand, d-Dimensional Rectangle Stabbing is W[1]-hard with respect to the parameter k for $d \geq 3$ [2]. On the other hand, two special cases of Rectangle Stabbing in two dimensions have been shown to be fixed-parameter tractable [2]: If each rectangle is intersected by at most b horizontal but arbitrarily many vertical lines or at most b vertical but arbitrarily many horizontal lines, or if each horizontal line intersects at most b rectangles, then RECTANGLE STABBING is fixed-parameter tractable with respect to the combined parameter b, d. The parameterized complexity of RECTANGLE STABBING without restrictions, however, remained open so far [2].

The contributions of this paper are the following. In Section 3, we settle the question of Dom and Sikdar [2] for the parameterized complexity of (2-DIMENSIONAL) RECTANGLE STABBING by proving its W[1]-hardness with respect to the parameter k as well as its membership in W[1]. Our proofs also show the W[1]-completeness of the more general problem 2-C1P-SET COVER (see Section 2), which was also open so far [2]. In Section 4, we consider the restriction of RECTANGLE STABBING where all rectangles in the input are squares of the same size that do not intersect. After showing its NP-hardness, we prove that this variant is fixed-parameter tractable. Due to the lack of space, some proofs are omitted.

2 Preliminaries

In the framework of parameterized complexity [3, 6, 17], the running time of an algorithm is viewed as a function of two quantities: the size of the given problem instance *and* a *parameter*. Thus, a parameterized problem is a subset of $\Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet and $\mathbb{N}$ is the set of positive integers; an instance of a parameterized problem is a pair (I, k), where k is called the parameter. A parameterized problem is said to be *fixed-parameter tractable (FPT)* with respect to the parameter k if there exists an algorithm for the problem running in $f(k) \cdot |I|^{O(1)}$ time, where f is a computable function only depending on k.

A parameterized problem π_1 is *fixed-parameter reducible* to a problem π_2 if there are two computable functions $f, g : \mathbb{N} \to \mathbb{N}$ and an algorithm that transforms an instance (I, k) of π_1 into an instance $(I', f(k))$ of π_2 in $g(k) \cdot |I|^{O(1)}$ time such that $(I', f(k))$ is a *yes*-instance for π_2 iff (I, k) is a *yes*-instance for π_1. The complexity hierarchy used for characterizing the hardness of parameterized problems is the W-*hierarchy* consisting of the classes $\mathrm{W}[1], \mathrm{W}[2], \ldots, \mathrm{W}[\mathrm{Sat}], \mathrm{W}[\mathrm{P}]$, interrelated as follows: $\mathrm{FPT} \subseteq \mathrm{W}[1] \subseteq \mathrm{W}[2] \subseteq \ldots \subseteq \mathrm{W}[\mathrm{Sat}] \subseteq \mathrm{W}[\mathrm{P}]$. There is strong evidence that all these subset inclusions are strict, which means that there are problems in W[1] that are presumably not fixed-parameter tractable and, in particular, that W[1]-*hard* problems are not fixed-parameter tractable [3, 6, 17]. To show that a problem is W[1]-hard (is in W[1]), one needs to exhibit a fixed-parameter reduction from a known W[1]-hard problem to the problem at hand (from the problem at hand to a problem known to be in W[1]). A problem is W[1]-*complete* if it is W[1]-hard and in W[1].

When considering an instance (L, R, k) of RECTANGLE STABBING, let $L = V \cup H$, where $V = \{v_1, \ldots, v_n\}$ are the vertical lines, ordered from left to right, and $H = \{h_1, \ldots, h_m\}$ are the horizontal lines, ordered from top to bottom. For a rectangle $r \in R$, let $\mathrm{lx}(r), \mathrm{rx}(r), \mathrm{tx}(r), \mathrm{bx}(r)$ be the indices of the leftmost, rightmost, topmost and bottommost line intersecting r. We define the

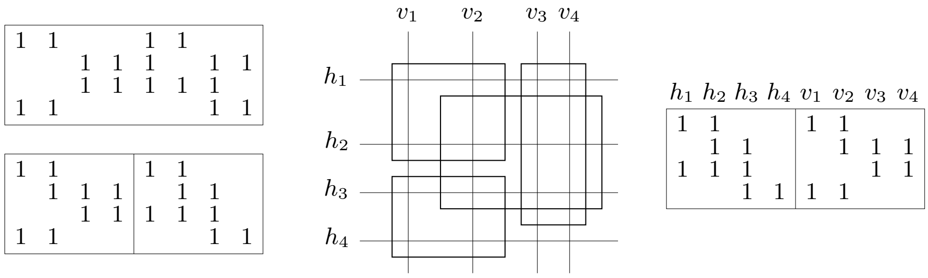

Fig. 1. Left top: A matrix having the 2-C1P, but not the 2-SC1P. Left bottom: A matrix having the 2-SC1P. Middle and right: Illustration of the equivalence between Rectangle Stabbing and 2-SC1P-Set Cover. In all figures of this paper, only the 1-entries of the matrices are displayed (that is, all 0-entries are omitted).

width $\text{wh}(r) := \text{rx}(r) - \text{lx}(r) + 1$ and the height $\text{ht}(r) := \text{bx}(r) - \text{tx}(r) + 1$ as the number of vertical and horizontal lines, respectively, intersecting r. A rectangle r is called a *square* if $\text{wh}(r) = \text{ht}(r)$.

All graphs that we consider are undirected; a graph $G = (V, E)$ is called *k-colorable* if there is a function $c : V \to \{1, \ldots, k\}$ satisfying $\forall \{u, v\} \in E : c(u) \neq c(v)$; the function c is then called a *k-coloring* for G.

For a simpler description of our algorithms and reductions, we will consider Rectangle Stabbing as a covering problem on binary matrices, which is a restricted version of the NP-complete problem Set Cover.[3]

Set Cover
Input: A binary matrix M and a positive integer k.
Question: Is there a set C' of at most k columns of M such that the submatrix M' of M that consists of these columns has at least one 1 in every row?

To introduce restricted versions of Set Cover, we need the following.

Definition 1. *1. Given a binary matrix M, a* block of 1s *in a row of M is a maximal set of consecutive 1-entries in this row.*

2. A binary matrix M has the d-consecutive-ones property (d-C1P) *if in every row of M there are at most d blocks of 1s.*

3. A binary matrix M with columns $c_1, \ldots, c_n$ has the separated d-consecutive-ones property (d-SC1P) *if the columns of M can be partitioned into d sets of consecutive columns $C^1 = \{c_1, \ldots, c_{j_1}\}$, $C^2 = \{c_{j_1+1}, \ldots, c_{j_2}\}$, $\ldots$, $C^d = \{c_{j_{d-1}+1}, \ldots, c_n\}$ such that for every $i \in \{1, \ldots, d\}$ the submatrix of M consisting of the columns of C^i has at most one block of 1s per row.*

See Fig. 1 for an illustration for the d-C1P and d-SC1P.

[3] Set Cover is usually defined as a subset selection problem; however, the equivalence of our definition and the more common definition of Set Cover as a subset problem can easily be seen by identifying columns with subsets and rows with elements to be covered.

If Set Cover is restricted by demanding that the input matrix M must have the d-C1P, then we call the resulting problem d-C1P-Set Cover; if M must have the d-SC1P, then we call the resulting problem d-SC1P-Set Cover. For an illustration of the following observation, see Fig. 1.

Observation 1 Rectangle Stabbing *and* 2-SC1P-Set Cover *are equivalent: There is a polynomial-time computable one-to-one mapping between instances of* Rectangle Stabbing *and instances of* 2-SC1P-Set Cover *that leaves the parameter k unchanged and maps yes-instances to yes-instances and no-instances to no-instances.*

3 W[1]-Completeness of Rectangle Stabbing

In this section, we prove that, for the parameter k, 2-SC1P-Set Cover is W[1]-hard and 2-C1P-Set Cover is in W[1], which implies the W[1]-completeness of Rectangle Stabbing.

3.1 W[1]-Hardness of Rectangle Stabbing

We give a fixed-parameter reduction from the problem Multicolored Clique defined below to 2-SC1P-Set Cover—the W[1]-hardness of Rectangle Stabbing then follows from the W[1]-hardness of Multicolored Clique [5] and the equivalence between 2-SC1P-Set Cover and Rectangle Stabbing.

Multicolored Clique

Input: An undirected k-colorable graph $G = (V, E)$, a positive integer k, and a k-coloring $c : V \to \{1, \ldots, k\}$ for G.

Question: Is there a size-k clique in G?

The basic scheme of the reduction. The basic approach of our reduction is similar to the one used in the W[1]-hardness proof for 3-SC1P-Set Cover [2]. However, due to the more restricted nature of 2-SC1P-Set Cover, the technical details are more involved.

We use the "Multicolored Clique reduction technique" [5], where the key idea is to use an alternative, equivalent formulation of Multicolored Clique: Given an undirected k-colorable graph $G = (V, E)$, a positive integer k, and a k-coloring $c : V \to \{1, \ldots, k\}$ for G, find a set $E' \subseteq E$ with $|E'| = \binom{k}{2}$ and a set $V' \subseteq V$ with $|V'| = k$ that satisfy the following three constraints:

- Constraint 1: For every unordered pair $\{a, b\}$ of colors from $\{1, \ldots, k\}$, the edge set E' contains an edge whose endpoints are colored with a and b.
- Constraint 2: For every color from $\{1, \ldots, k\}$, the vertex set V' contains a vertex of this color.
- Constraint 3: If E' contains an edge $\{u, v\}$, then V' contains the vertices u and v.

Clearly, this formulation of Multicolored Clique is equivalent to the original definition. Given an instance (G, k, c) of Multicolored Clique, we construct an equivalent instance (M, k') of 2-SC1P-Set Cover based on this alternative formulation. The standard approach for such a construction would be to create a matrix M with $|V| + |E|$ columns, one column for each vertex and each edge of G, and to set $k' = k + \binom{k}{2}$. The rows of M would have to be constructed in such a way that any solution C' for 2-SC1P-Set Cover on (M, k') corresponded to a solution (E', V') as described above for Multicolored Clique on the instance (G, k, c). That is, the rows of M would have to enforce that Constraints 1–3 are satisfied. The problem with this standard approach is that the resulting matrix M does not have the 2-SC1P. Therefore, we use a construction that is based on the same ideas, but involves more columns and rows. In order to obtain a matrix that has the 2-SC1P, we add not one, but *three* columns to M for every edge e in G. Hence, an instance (G, k, c) of Multicolored Clique is mapped to an instance (M, k'), where $k' = 3 \cdot \binom{k}{2} + k$.

To describe the details of M's construction, let the *color of an edge* $\{u, v\}$, denoted $d(\{u, v\})$, be the set of colors of its endpoints, that is, $d(\{u, v\}) := \{c(u), c(v)\}$. We assume that the edges $E = \{e_1, \ldots, e_{|E|}\}$ and vertices $V = \{v_1, \ldots, v_{|V|}\}$ of G are ordered in such a way that edges and vertices of the same color appear consecutively. For every edge color $\{a, b\}$, we define: $E_{\{a,b\}} := \{e \in E \mid d(e) = \{a, b\}\}$, $\text{first}(\{a, b\}) := \min\{p \in \{1, \ldots, |E|\} \mid d(e_p) = \{a, b\}\}$, and $\text{last}(\{a, b\}) := \max\{p \in \{1, \ldots, |E|\} \mid d(e_p) = \{a, b\}\}$. The details of the construction of M read as follows, see also Fig. 2.

The columns of M. The matrix M has $3 \cdot |E| + |V|$ columns, partitioned into two sets C^1 and C^2. The column set C^1 consists of two subsets of columns: a subset D^1 consisting of the columns $c^1_1, \ldots, c^1_{|E|}$, and a subset D^2 consisting of the columns $c^2_1, \ldots, c^2_{|E|}$. The column set C^2 also consists of two subsets of columns: the subset D^3 consisting of the columns $c^3_1, \ldots, c^3_{|V|}$, and the subset D^4 consisting of the columns $c^4_1, \ldots, c^4_{|E|}$.

These columns are ordered as follows in M. The leftmost $2 \cdot |E|$ columns of M are those from C^1, the remaining $|V| + |E|$ columns are those from C^2. The columns from C^1 are ordered in such a way that columns corresponding to edges of the same color appear consecutively. More precisely, for every edge color $\{a, b\}$, there are $2 \cdot |E_{\{a,b\}}|$ consecutive columns

$$c^1_{\text{first}(\{a,b\})}, \ldots, c^1_{\text{last}(\{a,b\})}, c^2_{\text{first}(\{a,b\})}, \ldots, c^2_{\text{last}(\{a,b\})}.$$

The columns from C^2 are ordered as follows: To the right of the columns from C^1, there are the $|V|$ columns $c^3_1, \ldots, c^3_{|V|}$. The rightmost $|E|$ columns of M, finally, are the columns $c^4_1, \ldots, c^4_{|E|}$. Intuitively speaking, every column $c^1_p \in C^1$, every column $c^2_p \in C^1$, and every column $c^4_p \in C^2$ one-to-one corresponds to the edge $e_p \in E$, and every column $c^3_q \in C^2$ one-to-one corresponds to the vertex $v_q \in V$.

The rows of M. The rows of M have to ensure that every solution C' for 2-SC1P-Set Cover on $(M, k' = 3 \cdot \binom{k}{2} + k)$ corresponds to a subset of edges and vertices of G satisfying Constraints 1–3. Since there are three columns in M for every edge in G, we need *four* types of rows:

	C^1										C^2													
	...	$\{\text{red}, \text{blue}\}$								...	...	red		...	blue			...	...	$\{\text{red}, \text{blue}\}$				...
	...	c_4^1	c_5^1	c_6^1	c_7^1	c_4^2	c_5^2	c_6^2	c_7^2	...	...	c_2^3	c_3^3	...	c_7^3	c_8^3	c_9^3	...	...	c_4^4	c_5^4	c_6^4	c_7^4	...
$r^1_{\{\text{red},\text{blue}\},D^1}$		1	1	1	1																			
$r^1_{\{\text{red},\text{blue}\},D^2}$						1	1	1	1															
$r^1_{\{\text{red},\text{blue}\},D^4}$																				1	1	1	1	
r^2_{red}												1	1											
r^2_{blue}															1	1	1							
$r^3_{\{\text{red},\text{blue}\},D^1,1}$		1																			1	1	1	
$r^3_{\{\text{red},\text{blue}\},D^1,2}$		1	1																			1	1	
$r^3_{\{\text{red},\text{blue}\},D^1,3}$		1	1	1																			1	
$r^3_{\{\text{red},\text{blue}\},D^1,4}$			1	1	1															1				
$r^3_{\{\text{red},\text{blue}\},D^1,5}$				1	1															1	1			
$r^3_{\{\text{red},\text{blue}\},D^1,6}$					1															1	1	1		
$r^3_{\{\text{red},\text{blue}\},D^2,1}$						1															1	1	1	
$r^3_{\{\text{red},\text{blue}\},D^2,2}$						1	1															1	1	
$r^3_{\{\text{red},\text{blue}\},D^2,3}$						1	1	1															1	
$r^3_{\{\text{red},\text{blue}\},D^2,4}$							1	1	1											1				
$r^3_{\{\text{red},\text{blue}\},D^2,5}$								1	1											1	1			
$r^3_{\{\text{red},\text{blue}\},D^2,6}$									1											1	1	1		
$r^4_{e_5,v_2}$				1	1	1						1												
$r^4_{e_5,v_8}$				1	1	1										1								
...																								

Fig. 2. Example for the construction of M in the W[1]-hardness proof for 2-SC1P-Set Cover. We assume that in G there are exactly two red vertices v_2, v_3 and exactly three blue vertices v_7, v_8, v_9, among vertices of other colors. Moreover, the only edges between red and blue vertices are e_4, e_5, e_6, e_7 with $e_5 = \{v_2, v_8\}$.

Rows of Type 1 and 2 ensure that any set of k' columns that forms a solution for 2-SC1P-Set Cover contains exactly $\binom{k}{2}$ columns from D^1—one of each edge color—, $\binom{k}{2}$ columns from D^2—one of each edge color—, $\binom{k}{2}$ columns from D^4—one of each edge color—, and k columns from D^3—one of each vertex color. Type-3 rows ensure that the columns chosen from D^1, D^2, and D^4 are consistent: if a solution contains the column c_j^1, then it must contain c_j^4, and *vice versa*; analogously, if a solution contains the column c_j^2, then it must contain c_j^4, and *vice versa*. Finally, Type-4 rows ensure that if a solution contains a column c_j^1 (and, due to the Type-3 rows, the column c_j^2) corresponding to an edge $e_j = \{u, v\}$, then it also contains the columns corresponding to the vertices u and v. See Fig. 2 for an illustration of the following construction details. The argument that the reduction works correctly is omitted.

Type-1 rows. For every edge color $\{a, b\}$, M contains three rows $r^1_{\{a,b\},D^1}$, $r^1_{\{a,b\},D^2}$, and $r^1_{\{a,b\},D^4}$.

For $x = 1, 2, 4$, the row $r^1_{\{a,b\},D^x}$ has a 1 in every column $c_j^x \in D^x$ with $d(e_j) = \{a, b\}$, and 0s in all other columns.

Type-2 rows. For every vertex color $a \in \{1, \ldots, k\}$, M contains a row r^2_a which has a 1 in every column $c^3_j \in D^3$ with $c(v_j) = a$, and 0s in all other columns.

Observe that the rows of the Types 1 and 2 together with the value of k' force every solution for 2-SC1P-Set Cover on (M, k') to contain *exactly one* column from each of D^1, D^2, and D^4 for every edge color and *exactly one* column from D^3 for every vertex color.

Type-3 rows. For every edge color $\{a, b\}$, M contains a set of $2 \cdot (|E_{\{a,b\}}| - 1)$ rows $r^3_{\{a,b\},D^1,i}$ and a set of $2 \cdot (|E_{\{a,b\}}| - 1)$ rows $r^3_{\{a,b\},D^2,i}$, where in both cases $1 \leq i \leq 2 \cdot (|E_{\{a,b\}}| - 1)$.

A row $r^3_{\{a,b\},D^x,i}$ with $x \in \{1, 2\}$ and $i \in \{1, \ldots, |E_{\{a,b\}}| - 1\}$ has a 1 in every column $c^x_j \in C^x$ with $d(e_j) = \{a, b\}$ and $j < \text{first}(\{a, b\}) + i$ and every column $c^4_j \in C^4$ with $d(e_j) = \{a, b\}$ and $j \geq \text{first}(\{a, b\}) + i$, and 0s in all other columns.

A row $r^3_{\{a,b\},D^x,i}$ with $i \in \{|E_{\{a,b\}}|, \ldots, 2 \cdot (|E_{\{a,b\}}| - 1)\}$ has a 1 in every column $c^x_j \in C^x$ with $d(e_j) = \{a, b\}$ and $j \geq \text{first}(\{a, b\}) + i - (|E_{\{a,b\}}| - 1)$ and every column $c^4_j \in C^4$ with $d(e_j) = \{a, b\}$ and $j < \text{first}(\{a, b\}) + i - (|E_{\{a,b\}}| - 1)$, and 0s in all other columns.

To see that the columns selected from D^x, $x \in \{1, 2\}$, and D^4 are consistent in every solution for 2-SC1P-Set Cover on (M, k'), observe that taking a column c^x_j into the solution, this column does not contain a 1 from the rows $r^3_{\{a,b\},D^x,j-\text{first}(\{a,b\})}$ and $r^3_{\{a,b\},j-\text{first}(\{a,b\})+|E_{\{a,b\}}|}$ (if existing). Hence, the single column from D^4 belonging to the solution must be c^4_j.

Type-4 rows. For every edge $e_p = \{v_{q_1}, v_{q_2}\} \in E$, the matrix M contains two rows $r^4_{e_p,v_{q_1}}$ and $r^4_{e_p,v_{q_2}}$.

For $i = 1, 2$, the row $r^4_{e_p,v_{q_i}}$ has a 1 in every column $c^1_j \in C^1$ with $d(e_j) = d(e_p)$ and $j > p$, every column $c^2_j \in C^2$ with $d(e_j) = d(e_p)$ and $j < p$, and the column $c^3_{q_i} \in C^3$, and 0s in all other columns.

Theorem 1. 2-SC1P-Set Cover*,* 2-C1P-Set Cover*, and* Rectangle Stabbing *are* W[1]*-hard with respect to the parameter k.*

Using a modification of the above construction, we can also show:

Theorem 2. *The restricted variant of* Rectangle Stabbing *where all rectangles in R are squares having the same width and the same height is* W[1]*-hard with respect to the parameter k.*

3.2 Membership of Rectangle Stabbing in W[1]

The membership of Rectangle Stabbing, and, more general, of 2-C1P-Set Cover, in W[1] can be shown in analogy to the proof given by Marx [15] for Dominating Set on intersection graphs of axis-parallel rectangles: One exhibits a fixed-parameter reduction to the W[1]-complete [3] problem Short Turing Machine Acceptance, defined as follows.

Short Turing Machine Acceptance
Input: The description of a nondeterministic Turing machine N and a positive integer k'.
Question: Can N stop within k' steps on the empty input string?

To reduce 2-C1P-Set Cover to Short Turing Machine Acceptance, one constructs, for a given instance (M, k) of 2-C1P-Set Cover, a nondeterministic Turing machine N that can stop after $k' = f(k)$ steps on the empty input string iff (M, k) is a *yes*-instance. Intuitively speaking, this Turing machine N nondeterministically decides in $f(k)$ steps whether the tuple (M, k), which is encoded into the internal states and the transition function of N, is a *yes*-instance of 2-C1P-Set Cover or not, and correspondingly it either stops after $f(k)$ steps or goes into an infinite loop (details are omitted).

Theorem 3. *For the parameter k,* 2-C1P-Set Cover*,* 2-SC1P-Set Cover *and* Rectangle Stabbing *are in* W[1].

4 Stabbing Nonoverlapping Squares of the Same Size

In this section, we consider the natural restriction of Rectangle Stabbing where no two rectangles from R "overlap". Two rectangles r_1, r_2 *overlap* if there exist a vertical line v and a horizontal line h that both intersect r_1 as well as r_2. Moreover, we further restrict the problem by demanding that all rectangles in R are squares of the same size, meaning that there is a number b such that for every rectangle r in R we have $\text{wh}(r) = \text{ht}(r) = b$. We call this restricted problem variant Disjoint b-Square Stabbing; it is equivalent to the stabbing problem where a set of unit squares in the plane is given (but no lines are given) and a minimum number of horizontal and vertical lines shall be inserted that stab all the squares.

Basic Observations. We show that, one the one hand, the problem Disjoint b-Square Stabbing is NP-complete for every $b \geq 2$, while, on the other hand, it is polynomial-time solvable for $b = 1$. To prove the NP-hardness, one can reduce from the NP-complete [7] problem Vertex Cover (details omitted).

Theorem 4. Disjoint b-Square Stabbing *is* NP*-complete for every $b \geq 2$.*

Complementing Theorem 4, one can see that Disjoint b-Square Stabbing is polynomial-time solvable for $b = 1$. To this end, observe that every instance of Disjoint 1-Square Stabbing is equivalent to an 2-SC1P-Set Cover instance (M, k) where the column set of M can be partitioned into two sets C^1 and C^2 of consecutive columns such that every row of M contains exactly one 1 in a column of C^1 and one 1 in a column of C^2. Such a matrix can be interpreted as a bipartite graph, and, hence, (M, k) corresponds to an instance of Vertex Cover on bipartite graphs. Vertex Cover on bipartite graphs, however, is known to be polynomial-time solvable.

Fixed-parameter tractability. We show that Disjoint b-Square Stabbing is in FPT with respect to the parameter k. To this end, we present a search-tree algorithm, that is, a recursive algorithm that in each recursive step branches into a bounded number of cases. More precisely, in each step the algorithm first determines a subset of the given lines in such a way that every size-k solution for the Rectangle Stabbing instance must contain at least one of these lines, and then recursively tests which of these lines leads to the desired solution.

In order to bound the size of the above subsets of lines and, thus, the number of cases to branch into, in each step a set of *data reduction rules* is applied. Each of these rules takes as input an instance (L, R, k) of Rectangle Stabbing and outputs in polynomial time an instance (L', R', k') of Rectangle Stabbing such that $|L'| \leq |L|$, $|R'| \leq |R|$, $k' \leq k$, and (L', R', k') is a *yes*-instance iff (L, R, k) is a *yes*-instance. An instance to which none of the rules can be applied is called *reduced*. Our data reduction rules read as follows.

Rule 1: If there are two lines $l_1, l_2 \in L$ such that every rectangle in R that is intersected by l_2 is also intersected by l_1, then delete l_2.
Rule 2: If there are two rectangles $r_1, r_2 \in R$ such that every line in L that intersects r_1 also intersects r_2, then delete r_2.
Rule 3: If there are $k+2$ rectangles $r_1, \ldots, r_{k+2} \in R$ such that no horizontal line intersects more than one of these rectangles and, for each $i \in \{1, \ldots, k+1\}$ we have $\text{lx}(r_i) \geq \text{lx}(r_{k+2})$ and $\text{rx}(r_i) \leq \text{rx}(r_{k+2})$, then delete r_{k+2}.

While the correctness of Rules 1 and 2 is obvious, the correctness of Rule 3 follows from the fact that k horizontal lines cannot intersect all rectangles $r_1, \ldots, r_{k+1}$. Note that the data reduction rules may transform an instance of Disjoint b-Square Stabbing into an instance of Rectangle Stabbing where the rectangles have heights or widths smaller than b; anyway, we call such an instance a *reduced instance of* Disjoint b-Square Stabbing.

The following observation is an immediate consequence of Rule 1.

Observation 2 *In a reduced problem instance of* Rectangle Stabbing*, for every vertical line $v_j \in V$, there exist rectangles $r, r' \in R$ with $\text{lx}(r) = j$ and $\text{rx}(r') = j$.*

Observation 3 *After applying any sequence of data reduction rules to an instance of* Disjoint b-Square Stabbing*, for any two rectangles $r_1, r_2 \in R$, we have $\text{lx}(r_1) > \text{lx}(r_2) \Rightarrow \text{rx}(r_1) \geq \text{rx}(r_2)$ and $\text{rx}(r_1) < \text{rx}(r_2) \Rightarrow \text{lx}(r_1) \leq \text{lx}(r_2)$.*

The latter observation follows from the fact that every instance of Disjoint b-Square Stabbing has the property described in the observation, and none of the data reduction rules destroys the property. The next observation follows directly from Rule 3; the proof of Lemma 1 is omitted.

Observation 4 *In a reduced instance of* Disjoint b-Square Stabbing*, for every $j \in \{1, \ldots, n\}$ there are at most $k+1$ rectangles r with $\text{lx}(r) = j$.*

Lemma 1. *For every rectangle r in a reduced instance, there are at most k rectangles r' with $\text{rx}(r') < \text{rx}(r)$ and $\text{lx}(r') \geq \text{lx}(r)$, and all these rectangles have $\text{lx}(r') = \text{lx}(r)$.*

Lemma 2. *Let r be a rectangle with $\mathrm{wh}(r) > xk + 1$ for $x \geq 2$ in a reduced instance. Then there exists a rectangle r' with $\mathrm{lx}(r') < \mathrm{lx}(r)$ and $(x-1)k + 1 < \mathrm{wh}(r') \leq xk + 1$.*

Proof. We show that the lemma is true if r has minimum $\mathrm{lx}(r)$ under all rectangles whose width is greater than $xk + 1$; this suffices to prove the lemma.

Observation 2 implies the existence of a rectangle r' with $\mathrm{rx}(r') = p$ for every $p \in \{\mathrm{lx}(r), \ldots, \mathrm{rx}(r) - 1\}$. Due to Lemma 1, at most k of these rectangles can have $\mathrm{lx}(r') \geq \mathrm{lx}(r)$, and, hence, there exists $p \in \{\mathrm{rx}(r) - k - 1, \ldots, \mathrm{rx}(r) - 1\}$ such that there is a rectangle r' with $\mathrm{rx}(r') = p$ and $\mathrm{lx}(r') < \mathrm{lx}(r)$. For the width of r' we have $\mathrm{wh}(r') = \mathrm{rx}(r') - \mathrm{lx}(r') + 1 \geq \mathrm{rx}(r) - k - 1 - \mathrm{lx}(r') + 1 > \mathrm{rx}(r) - k - 1 - \mathrm{lx}(r) + 1 = \mathrm{wh}(r) - k - 1$. Due to the selection of r, no rectangle r' with $\mathrm{lx}(r') < \mathrm{lx}(r)$ can have $\mathrm{wh}(r') > xk + 1$, and, hence, we have $(x-1)k + 1 < \mathrm{wh}(r') \leq xk + 1$. □

Lemma 3. *If a reduced instance contains a rectangle r with $\mathrm{wh}(r) > 2k + 1$, then there exists a rectangle r' with the following properties.*

1. *$k + 1 < \mathrm{wh}(r') \leq 2k + 1$.*
2. *There are at least k rectangles r'' with $\mathrm{rx}(r'') \in \{\mathrm{lx}(r'), \ldots, \mathrm{rx}(r') - 1\}$.*
3. *All rectangles r'' with $\mathrm{rx}(r'') \in \{\mathrm{lx}(r'), \ldots, \mathrm{rx}(r') - 1\}$ have $\mathrm{lx}(r'') \leq \mathrm{lx}(r')$.*
4. *All rectangles r'' with $\mathrm{rx}(r'') \in \{\mathrm{lx}(r'), \ldots, \mathrm{rx}(r') - 1\}$ have $\mathrm{wh}(r'') \leq 2k + 1$.*

Proof. The existence of a rectangle r' with $k + 1 < \mathrm{wh}(r') \leq 2k + 1$ follows from Lemma 2. Select a rectangle r' in such a way that $k + 1 < \mathrm{wh}(r') \leq 2k + 1$ and $\mathrm{lx}(r')$ is minimum under this property. Clearly, r' fulfills properties 2 and 3 because of Observations 2 and 3, respectively. Now assume, for the sake of a contradiction, that there is a rectangle r'' with $\mathrm{rx}(r'') \in \{\mathrm{lx}(r'), \ldots, \mathrm{rx}(r') - 1\}$ and $\mathrm{wh}(r'') > 2k + 1$. Clearly, we have $\mathrm{lx}(r'') < \mathrm{lx}(r')$. Together with Lemma 2, this implies the existence of a rectangle r''' that has $\mathrm{lx}(r''') < \mathrm{lx}(r'') < \mathrm{lx}(r')$ and $k + 1 < \mathrm{wh}(r''') \leq 2k + 1$, contradicting the selection of r'. □

Theorem 5. Disjoint b-Square Stabbing *is solvable in $(4k+1)^k \cdot n^{O(1)}$ time.*

Proof. If all rectangles have width at most $2k + 1$, the instance can be solved in $(2k + 2)^k \cdot n^{O(1)}$ time [2]. Otherwise, there is a rectangle r' as described in Lemma 3, such that the vertical line going through $\mathrm{lx}(r')$ intersects more than k rectangles whose width is at most $2k+1$. Since no horizontal line intersects more than one of these rectangles, not all of them can be stabbed by horizontal lines. Therefore, the solution must contain a vertical line intersecting at least two of these rectangles. There are at most $4k + 1$ such lines. □

5 Open Questions

Is Rectangle Stabbing in FPT when the input consists of nonoverlapping arbitrary rectangles? Is there a polynomial-size problem kernel for Disjoint b-Square Stabbing? Is d-Dimensional Rectangle Stabbing in W[1] when parameterized by both k *and* d?

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