Some New Developments in Parameterized Complexity

Mike Fellows, University of Newcastle

The talk will describe some surprising recent developments in the subject of parameterized complexity, after first reminding beginners in this area of some of the basic ideas and paradigms. In parameterized complexity the basic object of study is a parameterized problem, specified by: (1) the input to the problem, (2) the parameter — some aspect or part of the input that might reasonably be small (such as the nesting depth of a logical expression, or the size of the database query, or the number of biological sequences to be aligned, or the number of vertices in special set), and (3) the question to be answered.

As an example, the obvious parameterization of the VERTEX COVER problem takes as input a graph G and a positive integer k (with k considered to be the parameter) and asks whether G has a vertex cover consisting of k vertices. Although this problem is NP-complete, it has recently been shown to be solvable in time $O(n + (1.271)^k)$. Experimental results indicate that the problem is now well-solved for graphs of any size so long as the parameter k is less than around 400. This is a very desirable outcome for an NP-complete problem.

In contrast, for a similar obvious parameterization of the DOMINATING SET problem by the size of the dominating set of vertices, the best known algorithm is still $O(n^{k+1})$ by the brute force algorithm that simply tries all k-subsets. (Keep in mind that any algorithm with a polynomial exponent larger than 3 or 4 is pretty well useless.)

The contrast between these two outcomes is what parameterized complexity is all about. When the exponential difficulties of a problem can be separated from polynomial costs associated with the overall input size n, and confined to a function purely of the parameter (as in the above example of VERTEX COVER) then this is termed *fixed-parameter tractability* (fpt). One may view the resulting theory as effectively a two-dimensional generalization of polynomial time.

Not all parameterized problems appear to be fixed-parameter tractable. For example, we might consider the k-STEP HALTING PROBLEM for nondeterministic Turing machines (where the number of possible transitions possible at each step of the computation is unbounded, and thus may be as large as n, the size of the Turing machine description). Here k is the parameter, and the question is whether it is possible for the Turing machine to reach a halting configuration in at most k steps. This can obviously be solved by brute force in time $O(n^k)$ by exploring the depth-k n-branching tree of all possible length k computations exhaustively. (Compare the running time above for DOMINATING SET.)

Intuitively, we would not expect to be able to do much better than that for the k-STEP HALTING PROBLEM, because Turing machines are so opaque and amorphous. This sheds light on the DOMINATING SET problem because there is a combinatorial reduction (much like an NP-completeness reduction, but additionally preserving the parameter structure in an appropriate way) from the k-STEP HALTING PROBLEM to the DOMINATING SET problem, so that if the latter were fixed-parameter tractable, then so would be the former. In the terminology of parameterized complexity, DOMINATING SET is $W[1]$-hard. The analogy with NP-hardness is quite strong, since the classical theory of "P vs NP" is built around the POLYNOMIAL-TIME HALTING PROBLEM for nondeterministic Turing machines as the primordial hard problem.

Turning now to the exciting recent developments:

(1) In contrast to the $W[1]$-hardness of the general naturally parameterized DOMINATING SET problem, the restriction to planar graphs PLANAR DOMINATING SET is fixed-parameter tractable. The parameter function has recently been improved to 8^k. More remarkably, it has also been shown that the problem can be solved in time $O(2^{c\sqrt{k}} + n)$.

(2) Surprisingly powerful new techniques have been introduced by Liming Cai and David Juedes that show that the above result is in some sense optimal: any fpt algorithm for PLANAR DOMINATING SET with a running time of $O(2^{o(\sqrt{k})} + n^{O(1)})$ would imply FPT=$W[1]$.

(3) Associated with (2) are some striking new applications of parameterized complexity to the study of polynomial-time approximation algorithms.

In order to explain these applications we must first review a few more notions. One of the standard responses to NP-hardness is to look for polynomial-time approximation algorithms. For example, for the DOMINATING SET problem we might be quite happy with an algorithm that runs in time bounded by a polynomial $q(n, k)$ of n (the size of the graph) and $k = 1/\epsilon$, that produces a dominating set whose size is no more than $(1 + \epsilon)m$ where m is the minimum possible size of a dominating set for the graph. Such an algorithm is termed a polynomial-time approximation scheme (PTAS). The general DOMINATING SET problem does not admit a PTAS, but the PLANAR DOMINATING SET problem does.

There has been much recent work on devising PTAS's for various problems. A breakthrough result of Arora a few years ago was a PTAS for the Euclidean Traveling Salesman Problem, with running time $q(n, k) = O(n^{35k^2})$, a polynomial, but dire. There is an obvious parameter here: $k = 1/\epsilon$, and an issue that is essentially the same as in the discussion of VERTEX COVER and DOMINATING SET. An efficient PTAS (EPTAS) is a PTAS whose running time is fpt. Subsequent to the initial breakthrough on the Euclidean TSP problem by Arora, it has been shown that the Euclidean Traveling Salesman Problem has an EPTAS. However, for many other problems for which PTAS's are known, it is open whether these can be improved to EPTAS's.

Khanna and Motwani proposed that lurking behind "all" PTAS results was some sort of hidden or implicit planar structure. They defined several planar-structure-logic problems (that can be viewed as powerful generalizations of PLANAR DOMINATING SET), and showed that these have PTAS's — and showed that many PTAS results (for problems not about graphs and in no obvious way concerned with planarity) could be represented in these formalisms. It has been an open problem whether these very general planar-structure-logic problems that have this interesting general explanatory power about PTAS's admit EPTAS's. We have been able to settle this: these general problems are $W[1]$-hard for the parameter $k = 1/\epsilon$, and do not admit EPTAS's unless FPT=$W[1]$.