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A refined search tree technique for Dominating Set on planar graphs☆

Jochen Alber[a,1], Hongbing Fan[b], Michael R. Fellows[c,*], Henning Fernau[a,2], Rolf Niedermeier[a,3], Fran Rosamond[c], Ulrike Stege[d]

[a] *Universität Tübingen, Wilhelm-Schickard-Institut für Informatik, Sand 13, D-72076 Tübingen, Federal Republic of Germany*
[b] *Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, AB, Canada T1K 3M4*
[c] *School of Electrical Engineering and Computer Science, The University of Newcastle, University Drive, NSW 2308 Callaghan, Australia*
[d] *Department of Computer Science, University of Victoria, Victoria B.C., Canada V8W 3P6*

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Abstract

We establish a refined search tree technique for the parameterized DOMINATING SET problem on planar graphs. Here, we are given an undirected graph and we ask for a set of at most k vertices such that every other vertex has at least one neighbor in this set. We describe algorithms with running times $O(8^k n)$ and $O(8^k k + n^3)$, where n is the number of vertices in the graph, based on bounded search trees. We describe a set of polynomial time data-reduction rules for a more general "annotated" problem on black/white graphs that asks for a set of k vertices (black or white)

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* Corresponding author.

E-mail addresses: alber@informatik.uni-tuebingen.de (J. Alber), fan@cs.uleth.ca (H. Fan), mfellows@cs.newcastle.edu.au (Michael R. Fellows), fernau@informatik.uni-tuebingen.de (H. Fernau), niedermr@informatik.uni-tuebingen.de (R. Niedermeier), fran@cs.newcastle.edu.au (F. Rosamond), stege@cs.uvic.ca (U. Stege).

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[2] Also affiliated with The University of Newcastle.

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that dominate all the black vertices. An intricate argument based on the Euler formula then establishes an efficient branching strategy for reduced inputs to this problem. In addition, we give a family examples showing that the bound of the branching theorem is optimal with respect to our reduction rules. Our final search tree algorithm is easy to implement; its analysis, however, is involved.
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1. Introduction

Domination in graphs is considered to be among the most important problems in combinatorial optimization [23,24]. The problem remains NP-complete also when restricted to planar graphs [22]. From the viewpoint of polynomial-time approximation algorithms, however, the situation dramatically changes when going from general to planar graphs. Whereas the best approximation for general graphs (under some plausible complexity-theoretic assumptions) is $\Theta(\log n)$ [18], in case of planar graphs an efficient polynomial-time approximation scheme is known [7]. Somewhat analogously, we have a similar gap between Dominating Set on general graphs and planar graphs when shifting the focus to the parameterized complexity [15] of the problem, i.e., considering exact instead of approximate solutions. This issue, with a focus on search tree algorithms, is deeper explored in this paper.

The parameterized Dominating Set problem, where we are given an undirected graph $G = (V, E)$, a parameter k and ask for a set of vertices of size at most k that dominate all other vertices, is known to be $W[2]$-complete for general graphs [15]. The class $W[2]$ formalizes intractability from the point of view of parameterized complexity and $W[2]$-completeness indicates the impossibility of solving algorithms with running time $f(k)n^{O(1)}$ for some arbitrary, computable f only depending on k (i.e., no fixed-parameter tractability) [15]. By way of contrast, it is well-known that the problem restricted to planar graphs is fixed-parameter tractable. An algorithm running in time $O(11^k n)$ was claimed in [14,15]. The analysis of the algorithm there (which is nearly the same as our algorithm), turned out to be flawed. This paper seems to give the first completely correct analysis for Dominating Set on planar graphs with running time $O(c^k n)$ for *small* constant c (i.e., $c = 8$).

We mention that in companion work, algorithms with a running time of $O(c^{\sqrt{k}} n)$ for Dominating Set and related problems on planar graphs have been devised (see [3,5,6,19,21]). However, the huge worst-case constants c in these results may render them mainly of theoretical interest, although some empirical results indicate that they can work well in practice on some kinds of data [1].

1.1. Fixed-parameter algorithms based on search trees

A method that has proven to yield easy and powerful fixed-parameter algorithms is that of constructing a bounded search tree. Suppose, we are given a graph class $\mathcal{G}$ that is closed under taking subgraphs and that guarantees a vertex of degree d for some constant d. Such graph classes are, e.g., given by bounded degree graphs, or by graphs of bounded genus, and, hence, in particular, by planar graphs. More precisely, an easy computation (cf. [1]) shows that, e.g., the class $\mathcal{G}(S_g)$ of graphs that are embeddable on an orientable surface S_g of genus g guarantees a vertex of degree $d_g := \lceil 2(1 + \sqrt{3g + 1}) \rceil$ for $g > 0$, and, in case of planar graphs, $d_0 := 5$.

Consider the k-INDEPENDENT SET problem on $\mathcal{G}$, where, for given $G = (V, E) \in \mathcal{G}$, we seek for an independent set of size at least k. For a vertex u with degree at most d and neighbors $N(u) := \{u_1, \ldots, u_d\}$, we can choose one vertex $w \in N[u] := \{u, u_1, \ldots, u_d\}$ to be in an optimal independent set and continue the search on the graph G' where we deleted $N[w]$. This observation yields a simple $O((d+1)^k n)$ degree-branching search tree algorithm.

In the case of k-DOMINATING SET, the situation seems more intricate. Clearly, again, either u or one of its neighbors can be chosen to be in an optimal dominating set. However, removing u from the graph leaves all its neighbors being already dominated, but still also being suitable candidates for an optimal dominating set. This consideration leads us to formulate our search tree procedure in a more general setting, where there are two kinds of vertices in our graph. We stress this fact by partitioning the vertex set V of G into two disjoint sets B and W of *black* and *white* vertices, respectively, i.e., $V = B \uplus W$, where $\uplus$ denotes disjoint set union. We will also call this kind of graph a *black* and *white* graph.

ANNOTATED DOMINATING SET
Input: A black and white graph $G = (B \uplus W, E)$, and a positive integer k.
Parameter: k
Question: Is there a choice of at most k vertices $V' \subseteq V = B \uplus W$ such that, for every vertex $u \in B$, there is a vertex $u' \in N[u] \cap V'$? In other words, is there a set of at most k vertices (which may be either black or white) that dominates the set of black vertices?

In each step of the search tree, we would like to branch according to a low-degree black vertex. By our assumptions on the graph class, we can guarantee the existence of a vertex $u \in B \uplus W$ with $\deg(u) \leqslant d$. However, as long as *not all* vertices have degree bounded by d (as, e.g., the case for graphs of bounded genus g, where only *the existence* of a vertex of degree at most d_g is known), this vertex need not necessarily be black. These considerations show that a direct $O((d+1)^k n)$ search tree algorithm for DOMINATING SET seems out of reach for such graph classes.

1.2. Our results

In this paper, we present a fixed-parameter algorithm for (ANNOTATED) DOMINATING SET on planar graphs with running time $O(8^k n)$. For that purpose, we provide a set of reduction rules and, then, use a search tree in which we are constantly simplifying the instance according to the reduction rules (see Section 3.1). The branching in the search tree will be done with respect to low degree black vertices. The analysis of this algorithm is based on an intricate branching theorem (see Section 3.2) that depends ultimately on the Euler formula for planar graphs applied to reduced black/white graphs. In addition, we give a family of examples showing that the bound of the branching theorem is optimal (see Section 3.5), provided that no other reduction rules than those listed in Section 3.1 are employed. Finally, it is worth noting that the algorithm we present is very simple and easy to implement.

2. Preliminaries

We assume familiarity with basic notions and concepts in graph theory as presented, for example, in [13,28]. An undirected graph G is specified by a pair of sets (V, E), where V is the set of vertices of G and

E is the set of edges of G. Sometimes, we also write $V(G)$ and $E(G)$ in order to denote the vertex and edge set of G, respectively. For a graph $G = (V, E)$ and a vertex $u \in V$, we use $N(u)$ and $N[u]$, respectively, to denote the open and closed neighborhood of u, respectively. Hence, $N(u) = \{v \in V \mid \{u, v\} \in E\}$, and $N[u] = N(u) \cup \{u\}$. To avoid ambiguity, we sometimes write $N_G(u)$ and $N_G[u]$ to refer to the neighborhood in G. By $\deg_G(u) := |N_G(u)|$, we denote the *degree* of the vertex u in G. A *pendant* vertex is a vertex of degree one.

For $V' \subseteq V$, the subgraph induced by V' is denoted by $G[V']$. In particular, we use the abbreviation $G - V' := G[V \setminus V']$. If V' is a singleton, then we omit brackets and simply write $G - v$ for a vertex v. In addition, we write $G - e$ or $G + e$ when we delete or add an edge e to G without changing the vertex set of G.

Let G be a connected planar graph, i.e., a connected graph that admits a crossing-free embedding in the plane (i.e., a drawing in the plane without crossings). Such an embedding is called a *plane embedding*. A planar graph together with a plane embedding is called a *plane graph*. Note that a plane graph can be seen as a subset of the Euclidean plane $\mathbb{R}^2$. The set $\mathbb{R}^2 \setminus G$ is open; its regions are the *faces* of G. Let $\mathcal{F}$ be the set of faces of a plane graph. The *size of a face* $F \in \mathcal{F}$ is the number of vertices on the boundary of the face. A *triangular face* is a face of size three. If G is a plane graph and $V' \subseteq V$, then $G[V']$ and $G - V'$ can be always considered as plane graphs with an embedding inherited from the embedding of G.

3. The algorithm and its analysis

Our algorithm (Section 3.4) is based on reduction rules (see Section 3.1) and an improved branching theorem (see Sections 3.2 and 3.3 for proof details). With respect to our set of reduction rules, we show optimality for the branching theorem (Section 3.5).

3.1. Reduction rules

We consider the following reduction rules for simplifying the ANNOTATED DOMINATING SET problem on planar graphs. In developing the search tree, we will always assume that we are branching from a "reduced instance;" thus, we are constantly simplifying the instance according to the reduction rules given below (the details will be explained later).[4] When a vertex u is placed in the dominating set D by a reduction rule, then the target size k for D is reduced to $k - 1$ and the neighbors of u are whitened.

(R1) Delete an edge between white vertices.

(R2) Delete a pendant white vertex.

(R3) If there is a pendant black vertex w with neighbor u (either black or white), then delete w, place u in the dominating set, and lower k to $k - 1$.

(R4) If there is a white vertex u of degree 2, with two black neighbors u_1 and u_2 connected by an edge $\{u_1, u_2\}$, then delete u.

(R5) If there is a white vertex u of degree 2, with black neighbors u_1, u_3, and there is a black vertex u_2 and edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ in G, then delete u.

[4] The idea of doing so-called *rekernelizations* (i.e., repeated application of reduction rules) while constructing the search tree was proposed in [16,26] in a somewhat different context.

(R6) If there is a white vertex u of degree 2, with black neighbors u_1, u_3, and there is a white vertex u_2 and edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ in G, then delete u.

(R7) If there is a white vertex u of degree 3, with black neighbors u_1, u_2, u_3 for which the edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ are present in G (and possibly also $\{u_1, u_3\}$), then delete u.

Let us call a set of simplifying reduction rules of a certain problem *sound* if, whenever (G, k) is some problem instance and instance (G', k') is obtained from (G, k) by applying one of the reduction rules, then (G, k) has a solution iff (G', k') has a solution.

Lemma 1. *The reduction rules are sound.*

Proof. Let us consider the different reduction rules one by one. Let $G = (B \uplus W, E)$ denote the "original" black and white graph and $G' = (B' \uplus W', E')$ the graph obtained by applying once the corresponding reduction rule.

(R1) Clearly, $D \subseteq B \uplus W$ is a dominating set for G if and only if it is a dominating set for G'.

(R2) If $D \subseteq B \uplus W$ is a dominating set for G which contains a pendant white vertex u, then observe that $D' := (D \setminus \{u\}) \cup N(u)$ is also a dominating set for G. Furthermore, $D' \subseteq B \uplus W$ (with $u \notin D'$) is a dominating set for G if and only if it is a dominating set for G'.

(R3) If $D \subseteq B \uplus W$ is a dominating set for G which contains a pendant black vertex w, then observe that $D' := (D \setminus \{w\}) \cup N(w)$ is also a dominating set for G. Moreover, $D' \subseteq B \uplus W$ (with $w \notin D'$) is a dominating set for G if and only if $D' \setminus N(w)$ is a dominating set for G', since the vertices in $N(N(w)) \setminus \{w\}$ have been whitened.

(R4) If $D \subseteq B \uplus W$ is a dominating set for G which contains a white vertex u of degree two (as required) with two black neighbors u_1 and u_2 connected by an edge $\{u_1, u_2\}$, then observe that $D' := (D \setminus \{u\}) \cup \{u_1\}$ is also a dominating set for G. Furthermore, $D' \subseteq B \uplus W$ (with $u \notin D'$) is a dominating set for G if and only if it is a dominating set for G'.

(R5) If $D \subseteq B \uplus W$ is a dominating set for G which contains a white vertex u of degree two (as required) with black neighbors u_1, u_3, and there is a black vertex u_2 and edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$, then observe that $D' := (D \setminus \{u\}) \cup \{u_2\}$ is also a dominating set for G. Furthermore, $D' \subseteq B \uplus W$ (with $u \notin D'$) is a dominating set for G if and only if it is a dominating set for G'.

(R6) Analogous argument as for (R5) because the color of the intermediate vertex u_2 did not matter in the preceding argument.

(R7) Again, the argument for (R5) is valid here, as well. Observe that we need u_2 to be black now since, otherwise (in particular when the edge $\{u_1, u_3\}$ is present), it would be possibly better to put u_3 into the dominating set (instead of u or u_2). □

We say that G is a *reduced* graph if none of the above reduction rules can be applied to G. If none of the rules (R1), (R2), (R4)–(R7) are applicable to G, we term G *nearly reduced*.

Let $H := G[B]$ denote the (plane embedded) subgraph of G induced by the black vertices. Let F denote the set of faces of H. Say that a face $f \in F$ is *empty* if, in the plane embedding of G, it does not contain any white vertices.

Lemma 2. *Let $G = (B \uplus W, E)$ be a plane black and white graph. If G is (nearly) reduced, then the white vertices form an independent set and every triangular face of $G[B]$ is empty.*

Proof. The result easily follows from the reduction rules (R1), (R2), (R4), and (R7). □

Let us introduce a further notion which is important in order to bound the running time of our reduction algorithm.[5] To this end, we introduce the following variants of reduction rules (R5) and (R6):

(R5′) If there is a white vertex u of degree 2, with black neighbors u_1, u_3 such that u_1 *has at most seven neighbors that have degree at least 4*, and there exists a black vertex u_2 and edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ in G, then delete u.

(R6′) If there is a white vertex u of degree 2, with black neighbors u_1, u_3 such that u_1 *has at most seven neighbors that have degree at least 4*, and there exists a white vertex u_2 and edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ in G, then delete u.

We say that G is a *cautiously reduced* graph if (R1), (R2), (R4), (R5′), (R6′), and (R7) cannot be applied anymore to G. Observe that Lemma 2 is also valid for cautiously reduced graphs.

Lemma 3. *Applying reduction rules* (R1), (R2), (R4), (R5′), (R6′), *and* (R7), *a given planar black and white graph* $G = (B \uplus W, E)$ *can be transformed into a cautiously reduced graph* $G' = (B' \uplus W', E')$ *in time* $O(n)$, *where n is the number of vertices in G.*

Proof. (R1) and (R2) can be applied in linear time, since all edges between white vertices and all white vertices of degree one can be removed by two scans of the vertices of the graph. Observe that applying rule (R1) may trigger (R2), and applying (R1) and (R2) may trigger the other rules, but not vice versa.

In the case of reduction rule (R4), for each white vertex of degree two, we determine the neighbors u_1 and u_2 and ask the query whether $\{u_1, u_2\}$ is an edge in G. If this is the case, we remove u. In total we have to answer at most $O(n)$ queries of this form, which can be done in linear time by sorting the edges and the queries via radix sort.

In the case of rule (R7), for each white vertex of degree three, we determine the neighbors u_1, u_2, and u_3 and ask the three queries whether any of the sets $\{u_i, u_j\}$ $(1 \leqslant i, j \leqslant 3, i \neq j)$ is an edge in G. If two of these queries are answered positively, we remove u. In total we have at most $O(n)$ queries of this form, which can be answered in linear time.

The tricky part is with rules (R5′) and (R6′), because we need some sort of amortized analysis. For each white vertex of degree two, we determine the neighbors u_1 and u_3 of u and check whether one of these vertices has at most seven neighbors that are of degree at least 4. (Observe that, for fixed u, this can be done in constant time since we only need to determine the degree of u_1 and u_3 in the graph $G - \{ v \in V(G) : \deg_G(v) < 4 \}$. These degrees could have been determined in a preprocessing step in linear time.) If this is not the case (i.e., if both vertices u_1 and u_3 have more than seven neighbors of degree at least 4), we leave the graph unchanged, since we aim at a cautiously reduced instance. Otherwise (assuming, w.l.o.g, that u_1 has at most seven such neighbors) we have to check whether u_1 is connected to u_3 by a vertex v. To answer this, we ask for each of the at most seven such neighbors v, the queries whether $\{v, u_3\}$ is an edge in G. If one such query is answered positively, we remove u. In total we have at most $O(n)$ queries of this form, which can be answered in linear time. It remains to check, whether u_1 is connected to u_3 by a vertex v of degree two or three. To cover these cases, we check for each vertex v of degree two or three, whether there are two neighbors u_1 and u_3 of v (among which one vertex has to

[5] In the conference version of this paper, it was stated that a graph can be reduced with respect to all rules in linear time. Thanks to Torben Hagerup (Augsburg) who pointed out a gap in our earlier argument and suggested the fix we here employ.

have at most seven neighbors of degree at least 4 and) which are connected by a white vertex of degree two. This requires at most three queries per vertex *v*. Therefore the total number of queries again is linear and, hence, can be answered in linear time. □

3.2. A new branching theorem

In the course of this section, we will prove the following main theorem of this paper.

Theorem 3.1. *If $G = (B \uplus W, E)$ is a planar black and white graph that is nearly reduced, then there exists a black vertex $u \in B$ with $\deg_G(u) \leqslant 7$.*

Since the proof of Theorem 3.1 is very technical, let us first give a brief overview of it. In Lemma 4, we specialize Euler's well-known formula for planar graphs to planar black and white graphs. This is a core tool within the proof of Theorem 3.1, which is done by contradiction. Lemma 5 sets up some additional information used in the proof of Theorem 3.1. Some additional technical notations are then introduced to simplify the statements and proofs of some more technical lemmas and propositions (exhibited in Section 3.3) on which the proof of Theorem 3.1 relies, and which are already used within this subsection.

The following technical lemma, based on an "Euler argument," will be needed.

Lemma 4. *Suppose $G = (B \uplus W, E)$ is a nearly reduced connected plane black and white graph with b black vertices, w white vertices, and e edges. Let the subgraph induced by the black vertices be denoted $H = G[B]$. Let c_H denote the number of connected components of H and let f_H denote the number of faces of H. Let*

$$z = (3(b + w) - 6) - e \tag{1}$$

measure the extent to which G fails to be a triangulation of the plane. If the criterion

$$3w - 4b - z + f_H - c_H < 7 \tag{2}$$

is satisfied, then there exists a black vertex $u \in B$ with $\deg_G(u) \leqslant 7$.

Proof. Let the (total) numbers of vertices, edges, and faces of *G* be denoted v, e, f, respectively. Let e_{bw} be the number of edges in *G* between black and white, and let e_{bb} denote the number of edges between black and black. With this notation, we have the following relationships.

$$v - e + f = 2 \qquad \text{(Euler formula for } G\text{)}, \tag{3}$$

$$v = b + w, \tag{4}$$

$$e = e_{bb} + e_{bw}, \tag{5}$$

$$b - e_{bb} + f_H = 1 + c_H \qquad \text{((extended) Euler formula for } H\text{)}, \tag{6}$$

$$2v - 4 - z = f \qquad \text{(by Eqs. (1), (3), and (4))}, \tag{7}$$

If the lemma were false, then the minimum degree would be at least eight. Hence, we would have, using (5),

$$8b \leqslant 2e_{bb} + e_{bw} = e_{bb} + e. \tag{8}$$

We will assume this and derive a contradiction. The following inequality holds:

$$\begin{aligned} 3 + c_H &= v + b - (e_{bb} + e) + f + f_H && \text{(by (3) and (6))}, \\ &\leqslant v + b - 8b + f + f_H && \text{(by (8))}, \\ &= 3v - 7b + f_H - 4 - z && \text{(by (7))}, \\ &= 3w - 4b + f_H - 4 - z && \text{(by (4))}. \end{aligned}$$

This yields a contradiction to (2). □

We will prove Theorem 3.1 by contradiction. The reduction rules give us additional helpful properties of an assumed counterexample. This is stated in the following lemma.

Lemma 5. *If there is any counterexample to Theorem 3.1, then there is a connected counterexample where* $\deg_G(u) \geqslant 3$ *for all* $u \in W$.

Proof. Suppose G is a counterexample to the theorem. Since all connected components of G will then also provide counterexamples, we can—w.l.o.g.—assume that G is connected. Then, G does not have any white vertices of degree 1, else reduction rule (R2) can be applied. Let G' be obtained from G by simultaneously replacing every white vertex u of degree 2 with neighbors x and y by an edge $\{x, y\}$. The neighbors x and y of u are necessarily black, else (R1) can be applied, and in each case the edge $\{x, y\}$ is not already present in G, else rule (R4) would apply. We argue that G' is nearly reduced. If not, then the only possibility is that reduction rule (R7) applies to some white vertex u of degree 3 in G'. If rule (R7) did not apply to u in G, then one of the edges between the neighbors of u must have been created in our derivation of G' from G, i.e., one of these edges replaced a white vertex u' of degree 2. But this implies that reduction rule (R6) could be applied in G to u', contradicting that G is nearly reduced. □

Before giving the proof of Theorem 3.1, we introduce the following notation:

Notation: Let $G = (B \uplus W, E)$ be a plane black and white graph and let $\mathcal{F}$ be the set of faces of $G[B]$ (not of G). Then, for each $F \in \mathcal{F}$, we let

- w_F denote the number of white vertices embedded in F,
- z_F denote the number of edges that would have to be added in order to complete a triangulation of that part of the embedding of G contained in F,
- t_F denote the number of edges needed to triangulate F in $G[B]$ (that is, triangulating only between the black vertices on the boundary of F, and noting that the boundary of F may not be connected), and
- c_F denote the number of connected components of the boundary of F, minus 1.

Proof of Theorem 3.1. We can assume that if there is a counterexample G then G is connected (Lemma 5), but the black subgraph $H := G[B]$ might not be connected. Moreover, by Lemma 5 we may assume that $\deg_G(u) \geqslant 3$ for all $u \in W$. If c_H denotes the number of components of H, by induction on c_H, it is easy to see that

$$c_H - 1 = \sum_{F \in \mathcal{F}} c_F.$$

Also, if z is the number of edges needed to triangulate G, we clearly get

$$z = \sum_{F \in \mathcal{F}} z_F.$$

Criterion (2) from Lemma 4 can be rephrased as

$$3 \sum_{F \in \mathcal{F}} w_F - \sum_{F \in \mathcal{F}} z_F - 4b + f_H - c_H < 7,$$

which is equivalent to

$$3 \sum_{F \in \mathcal{F}} (w_F + c_F/3 - z_F/3 + 1/3) - 4b - 2c_H < 6.$$

Now, assume that we can show the inequality

$$w_F + c_F/3 - z_F/3 + 1/3 \leqslant \alpha t_F + \beta \tag{9}$$

for some constants α and β (which will be determined later) and for every face F of the subgraph H. Call this our *linear bound* assumption. Then, criterion (2) will hold if

$$3 \sum_{F \in \mathcal{F}} (\alpha t_F + \beta) - 4b - 2c_H = \left(3\alpha \sum_{F \in \mathcal{F}} t_F\right) + \left(3\beta \sum_{F \in \mathcal{F}} 1\right) - 4b - 2c_H < 6.$$

Noting that $\sum_{F \in \mathcal{F}} t_F$ is the number of edges needed to triangulate H, we have

$$\sum_{F \in \mathcal{F}} t_F = 3b - 6 - e_{bb}.$$

The number of faces of H is $\sum_{F \in \mathcal{F}} 1 = f_H = e_{bb} - b + 1 + c_H$, by Euler's formula (6). Together, these give us the following targeted criterion:

$$3\alpha(3b - 6 - e_{bb}) + 3\beta(e_{bb} - b + 1 + c_H) - 4b - 2c_H < 6.$$

Multiplying out and gathering terms, we need to establish (using the linear bound assumption) that

$$b(9\alpha - 3\beta - 4) + e_{bb}(3\beta - 3\alpha) + 3\beta(1 + c_H) - 18\alpha - 2c_H < 6.$$

This inequality is easily verified for $\alpha = \beta = 2/3$.

To complete the argument, we need to establish that our linear bound assumption (9) with $\alpha = \beta = 2/3$ holds for faces of nearly reduced graphs, i.e., that

$$w_F + c_F/3 - z_F/3 \leqslant 2t_F/3 + 1/3. \tag{10}$$

But this is a consequence of the following Propositions 3.3.1 and 3.3.2. □

3.3. Proving the correctness of Eq. (10)

Lemma 6. *Let $G = (V_1 \uplus V_2, E)$ be a plane graph, where both $G[V_1]$ and $G[V_2]$ are forests. Then, we can add edges between some of the vertices of V_1 (yielding a graph $\tilde{G}$) so that $\tilde{G}[V_1]$ is a tree and $\tilde{G}$ is (also) a plane graph. The number of added edges equals the number of components of $G[V_1]$ minus one.*

Proof. We construct a tree connecting the V_1-vertices among themselves by recursively decrementing the number of components in $G[V_1]$ from $|V_1|$ to 1 by adding edges. This means that we are going to prove the lemma by induction over the number of components of $G[V_1]$. The induction base—where the number of these components equals one—trivially holds. In the induction step, we use the following claim.

Claim. *Let $G = (V_1 \uplus V_2, E)$ be a plane graph, where V_1 is an independent set in G and where $G[V_2]$ is a forest. Then, for every vertex $v \in V_1$, there exists another vertex $v' \in V_1$ such that the edge $\{v, v'\}$ can be additionally drawn in the embedded graph G without destroying planarity.*

Assume that the claim has been verified and that the assertion of the lemma holds for all graphs where $G[V_1]$ is a forest with c components. Consider now a graph G which satisfies the assumptions of this lemma and where $G[V_1]$ is a forest with $c+1$ components. Let the graph $G' = (V_1' \uplus V_2, E')$ be obtained from G by contracting all components of $G[V_1]$ to single vertices. Then, G' satisfies the assumption of the claim. Hence, a vertex can be drawn connecting two vertices u and u' in V_1' which represent components K and K' in G. Clearly, the edge e obtained by the claim can be drawn between two arbitrary vertices v and v' belonging to components K and K', respectively. Now, the induction hypothesis can be applied to $\hat{G} = G + e$, since $\hat{G}$ has only c components.

Proof of the Claim. Take some vertex $v \in V_1$. If there is no cycle enclosing v, it is possible to connect v with any other vertex in V_1 without destroying planarity. Otherwise, consider the set of all embedded cycles which enclose v. This set is partially ordered by the relation “cycle C_1 contains cycle C_2.” Take the smallest of these cycles. Since $G[V_2]$ is acyclic by assumption, this cycle must contain at least one vertex v' from V_1. By construction, an edge can be drawn between v and v' without destroying planarity. □

Proposition 3.3.1. *Let $G = (B \uplus W, E)$ be a nearly reduced plane black and white graph and let F be a face of $G[B]$. Then, using the notation introduced above, we have*

$$w_F + c_F \leqslant z_F + 1.$$

Proof. Consider the "face-graph" $G_F := G[B_F \cup W_F]$, where B_F is the set of black vertices forming the boundary of F and W_F is the set of white vertices inside F. Note that G_F may consist of several "black components," connected only to white vertices. Contracting each of these black components into one (black) vertex, we obtain the *bipartite* black and white graph G'_F. Note that both the black and also the white vertices form independent sets in G'_F by the above construction, since G is assumed to be nearly reduced. Clearly, G'_F is still planar. Since G'_F is a bipartite planar graph, the assumptions of Lemma 6 are fulfilled (with V_1 being the white vertices and V_2 being the black vertices) and we can connect the white vertices by a forest of $w_F - 1$ white–white edges. Observe that the resulting black and white graph G' again satisfies the assumptions of Lemma 6 (now, V_1 are the black vertices and V_2 are the vertices that induce a tree in G'). Thus, in addition, we can connect the black vertices among themselves by a tree of c_F black–black edges. Clearly, this implies that we can also add at least $c_F + w_F - 1$ new edges to G_F without destroying planarity. Hence, we need at least $c_F + w_F - 1$ additional edges to triangulate the interior of F in the graph G. □

The following technically involved lemma is used as induction base in the proof of Proposition 3.3.2 which completes the proof of Theorem 3.1.

Lemma 7. *Suppose $G = (B \uplus W, E)$ is a nearly reduced plane black and white graph, with $\deg_G(u) = 3$ for all $u \in W$. Let F be a face of $G[B]$. Then, using the notation introduced above, we have $w_F \leqslant t_F$.*

Proof. Let us consider a fixed embedding of the graph G in the plane, and consider a face F of the black induced subgraph $G[B]$. Let $W_F \subseteq W$ be the set of white vertices in the interior of F, and let $B_F \subseteq B$ denote the black vertices on the boundary of F. We want to find at least $|W_F|$ many black–black edges that can be added to $G[B]$ inside F without destroying planarity. For that purpose, define the set

$$E^{\text{poss}} := \{e = \{b_1, b_2\} \mid b_1, b_2 \in B_F \ \wedge \ e \notin E(G[B])\}$$

of pairs of black vertices that are not connected by an edge.

For a subset $W' \subseteq W_F$, we construct a bipartite graph

$$H(W') := (W' \uplus T(W'), E(W'))$$

as follows. In $H(W')$, the first bipartition set is formed by the vertices W' and the second one is given by the set

$$T(W') := \{e = \{b_1, b_2\} \in E^{\text{poss}} \mid \exists u \in W' : e \subseteq N_G(u)\}.$$

The edges in $H(W')$ are then given by

$$E(W') := \{\{u, e\} \mid u \in W', e \in T(W'), e \subseteq N_G(u)\}.$$

In this way, the set $T(W')$ gives us vertices in $H(W')$ that correspond to pairs $e = \{b_1, b_2\}$ of black vertices in B_F between which we still can draw an edge in $G[B]$. Note that the edge e can even be drawn in the interior of F, since b_1 and b_2 are connected by a white vertex in $W' \subseteq W_F$ and since each white vertex has degree three by assumption. In particular, this means that

$$|T(W_F)| \leqslant t_F. \tag{11}$$

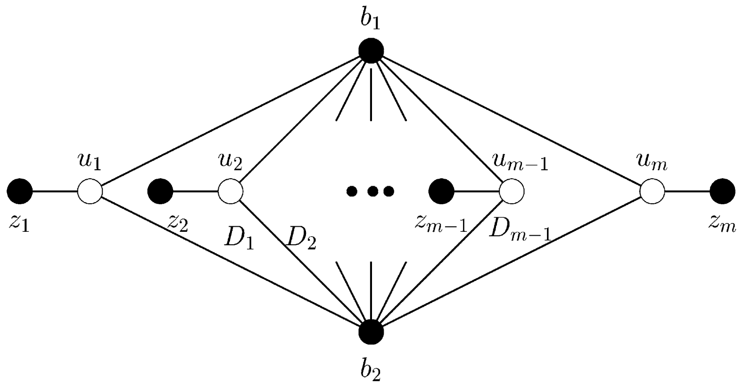

Fig. 1. Illustration of a diamond D generated by a pair vertices $\{b_1, b_2\} \in T(W_F)$.

Also, observe that, due to reduction rule (R7), for each $u \in W_F$, the neighbors $N(u) \subseteq B_F$ are connected by at most one edge in $G[B]$. By construction of $H(W_F)$, this means that

$$\deg_{H(W_F)}(u) \geqslant 2 \quad \text{for all} \quad u \in W_F. \tag{12}$$

The degree $\deg_{H(W_F)}(e)$ for an element $e = \{b_1, b_2\} \in T(W_F)$ tells us how many white vertices share the pair $\{b_1, b_2\}$ as common neighbors. We do case analysis according to this degree.

Case 1: Suppose that $\deg_{H(W_F)}(e) \leqslant 2$ for all $e \in T(W_F)$. Then, $H(W_F)$ is a bipartite graph, in which the first bipartition set has degree at least two (see Eq. (12)) and the second bipartition set has degree at most two. In this way, the second set cannot be smaller, which, using inequality (11), yields

$$w_F = |W_F| \leqslant |T(W_F)| \leqslant t_F.$$

Case 2: There exist elements $e = \{b_1, b_2\}$ in $T(W_F)$ which are shared as common neighbors by more than 2 white vertices (i.e., $\deg_{H(W_F)}(e) = m > 2$). Suppose that we have $u_1, \ldots, u_m \in W_F$ with $N_G(u_i) = \{b_1, b_2, z_i\}$ (i.e., $\{u_i, e\} \in E(W_F)$). We may assume that the vertices are ordered such that the closed region D bounded by $\{b_1, u_1, b_2, u_m\}$ contains all other vertices $u_2, \ldots, u_{m-1}$ (see Fig. 1).

We call D the *diamond* generated by $\{b_1, b_2\}$. Note that D consists of $m - 1$ regions, which we call *blocks* in the following; the block D_i is bounded by $\{b_1, u_i, b_2, u_{i+1}\}$ ($i = 1, \ldots, m-1$). Let $W_i \subseteq W_F$, and $B_i \subseteq B_F$, respectively, denote the white and black, respectively, vertices that lie in D_i. For the boundary vertices $\{b_1, b_2, u_1, \ldots, u_m\}$, we use the following convention: b_1, b_2 are added to all blocks, i.e., $b_1, b_2 \in B_i$ for all i; and u_i is added to the region where its third neighbor z_i lies in. A block is called *empty* if $B_i = \{b_1, b_2\}$ and, hence, $W_i = \emptyset$. Moreover, let $W_D := \bigcup_{i=1}^{m-1} W_i$ and $B_D := \bigcup_{i=1}^{m-1} B_i$.

We only consider diamonds where z_1 and z_m are not contained in D (see Fig. 1). The other cases can be treated with similar arguments.

Note that each block of a diamond D may contain further diamonds, the blocks of which may contain further diamonds, and so on. Since no diamonds overlap, the topological inclusion forms a natural ordering on the set of diamonds and their blocks.

We now use the following claim.

Claim. *For each diamond D generated by* $\{b_1, b_2\}$, *we can add* t_D (*where* $t_D \geqslant |W_D|$) *many black–black edges to* $G[B]$ *other than* $\{b_1, b_2\}$. *All of these additional edges can be drawn inside D and we still have the possibility to draw the edge* $\{b_1, b_2\}$.

Using this claim, we can finish the proof of Lemma 7: Consider all diamonds $D^1, \ldots, D^r$ which are not contained in any further diamond. Suppose D^i has boundary $\{b_1^i, u_1^i, b_2^i, u_{m_i}^i\}$ with $b_1^i, b_2^i \in B_F$ and $u_1^i, u_{m_i}^i \in W_F$. Let

$$W'_F := W_F \setminus \left(\bigcup_{i=1}^{r} W_{D^i} \right).$$

According to the claim, we already found $\sum_{i=1}^{r} t_{D^i}$ many black–black edges in E^{poss} inside the diamonds D^i. Observe that each pair $e^i = \{b_1^i, b_2^i\}$ is only shared as common neighbors by at most two white vertices (namely, u_1^i and $u_{m_i}^i$) in (sic!) W'_F. Hence, the bipartite graph $H(W'_F)$ again has the property that

- $\deg_{H(W'_F)}(e) \leqslant 2$ for all $e \in T(W'_F)$ and still
- $\deg_{H(W'_F)}(u) \geqslant 2$ for all $u \in W'_F$.[6]

Similar to Case 1 this proves that—additionally—we find t' (with $t' \geqslant |W'_F|$) many edges in E^{poss}. Hence,

$$w_F = |W_F| = |W'_F| + \left| \bigcup_{i=1}^{r} W_{D^i} \right| \leqslant t' + \left(\sum_{i=1}^{r} t_{D^i} \right) \leqslant t_F.$$

Proof of the Claim. We give an inductive argument proceeding from the "innermost" diamonds to the outer ones with respect to the inclusion ordering mentioned above.

Induction base: Consider an innermost diamond D with its blocks $D_1, \ldots, D_{m-1}$. We give a proof for the claim in the case where z_1 and z_m are not contained in D (see Fig. 1). The other cases work similarly. Suppose that there are $m' \leqslant m - 1$ many non-empty blocks. For each non-empty block, we consider the bipartite graph $H(W_i)$. Since D_i has no further diamonds in its interior, we again have the property that $\deg_{H(W_i)}(e) \leqslant 2$ for all $e \in T(W_i)$. This shows that $|W_i| \leqslant |T(W_i)|$ (with the same arguments as in Case 1). Note that all edges $e \in T(W_i)$ can be drawn in the interior of D_i. However, we might have used $\{b_1, b_2\}$ for each non-empty D_i, i.e., at most m' times. Since (according to the claim) we do not wish to use the edge $\{b_1, b_2\}$ at all, we use a set of m' many additional black–black edges from E^{poss} instead. These can be found as follows: From each z_i ($i = 1, \ldots, m - 1$) we can find an additional black–black edge to a black vertex in either D_i (if $z_i \notin B_i$) or D_{i-1} (if $z_i \in B_i$).[7] An easy analysis shows that this gives m' many additional edges.

Induction step: Consider a diamond D generated by $\{b_1, b_2\}$ with blocks $D_1, \ldots, D_m$ and suppose that, for all further diamonds inside the blocks D_i, the claim already holds true. Suppose we had "inner diamonds" $D_i^1, \ldots, D_i^{j_i}$ inside D_i. For the vertices $\bigcup_{\ell=1}^{j_i} W_{D_i^\ell}$, the induction hypothesis already

[6] Note that according to the claim the edges $\{b_1^i, b_2^i\}$ still can be used.

[7] If D_i (if $z_i \notin B_i$) or D_{i-1} (if $z_i \in B_i$) is empty, a black–black edge can be drawn directly from z_i to z_{i+1} or z_{i-1}.

assures that we find at least $\sum_{\ell=1}^{j_i} |W_{D_i^\ell}|$ many black–black edges from E^{poss} inside the diamonds $D_i^1, \ldots D_i^{j_i}$. Hence, it remains to consider $W_i' := W_i \setminus (\bigcup_{\ell=1}^{j_i} W_{D_i^\ell})$. The graph $H(W_i')$ has the properties that

$$\deg_{H(W_i')}(u) \geqslant 2 \quad \text{for all} \quad u \in W_i'$$

and that

$$\deg_{H(W_F')}(e) \leqslant 2 \quad \text{for all} \quad e \in T(W_i').$$

This means that we can argue in a manner similar to the induction base to see that we can find at least $\sum_{i=1}^m |W_i'|$ many additional black–black edges inside D not using the edge $\{b_1, b_2\}$. In total this gives us at least

$$\sum_{i=1}^{m} \left(|W_i'| + \sum_{\ell=1}^{j_i} |W_{D_i^\ell}| \right) = |W_D|$$

many edges. □

We show in the following proposition that the assumption that $\deg_G(u) = 3$ for all $u \in W_F$ is no restriction.

Remark 1. If F_1 and F_2 are two faces of $G[B]$ with common boundary edge e, then $t_{F_1} + t_{F_2} + 1$ equals t_F, where we now consider $(G - e)[B]$, and F is the face which results from merging F_1 and F_2 when deleting e.

Proposition 3.3.2. *Suppose $G = (B \uplus W, E)$ is a nearly reduced plane black and white graph, with $\deg_G(u) \geqslant 3$ for all $u \in W$. Let F be a face of $G[B]$. Then, using the notation introduced above, we have*

$$w_F \leqslant t_F.$$

Proof. Consider a nearly reduced black and white graph $G = (B \uplus W, E)$ with $\deg_G(u) \geqslant 3$ for all $u \in W$. If there is some $u \in W$ with $\deg_G(u) > 4$, then delete arbitrarily all except four edges incident with u. The black induced subgraph is preserved, and the resulting graph is still nearly reduced, since no rules apply to white degree-4-vertices. We can therefore assume without loss of generality that all white vertices of G have maximum degree four.

We will now show the claim by induction on the number $\#_4(W)$ of white vertices of degree four. Lemma 7 can be taken as the induction base. Assume the claim holds for each graph with $\#_4(W) \leqslant \ell$ and consider now the case that G has $\ell + 1$ white degree-4-vertices. Choose some arbitrary $u \in W$ with $\deg_G(u) = 4$. Let $\{b_1, \ldots, b_4\}$ be the neighbors of u in clockwise order. Because of planarity, we may assume further that $\{b_1, b_3\} \notin E$. Consider $G' = (G - u) + \{b_1, b_3\}$. We will argue that G' (or $G'' = (G - u) + \{b_2, b_4\}$ in one special case) is nearly reduced. This means that the induction hypothesis applies to G'. Hence, $w_F \leqslant t_F$ for all faces in $G'[B]$. Observe that G' contains all the faces of G except for the face F of G

which contains u; F might be replaced by two faces F_1 and F_2 with common boundary edge $\{b_1, b_3\}$. In this case, $w_{F_1} \leqslant t_{F_1}$, $w_{F_2} \leqslant t_{F_2}$, $w_{F_1} + w_{F_2} + 1 = w_F$ and, by Remark 1, $t_{F_1} + t_{F_2} + 1 = t_F$. Hence, $w_F \leqslant t_F$ by induction. In the case where face F still exists in G', it is trivial to see that $w_F \leqslant t_F$.

To complete the proof, we argue that G' has to be nearly reduced, in particular with respect to (R7). This is clear if $\forall b_i, \forall v \in N(b_i)$, $\deg_{G'}(v) = 4$, since no reduction rules apply to degree-4-vertices. We now address the possibility that u has degree-3-vertices as neighbors.

1. If a degree-3-vertex v is a neighbor of some b_i, but not of any b_j, $j \neq i$, then (R7) will not apply to v in G', if it has not been applicable to v in G already.
2. Consider the case that a degree-3-vertex v is a neighbor of two $b_i, b_j, i \neq j$. If $|\{i, j\} \cap \{1, 3\}| \leqslant 1$, then introducing the edge $\{b_1, b_3\}$ will not add any further edge to $N(v)$. Hence, (R7) will not be applicable to v in G' unless we could have applied this rule already in G. If $\{i, j\} = \{1, 3\}$, then, by planarity, $\{b_2, b_4\} \notin E(G)$ and we could consider $G'' = (G - u) + \{b_2, b_4\}$ instead of G' with an argument similar to the case $\{i, j\} = \{2, 4\}$.
3. If a degree-3-vertex v is a neighbor of three distinct b_i, b_j, b_k, then we can argue in a manner similar to 2.

This concludes the proof of the proposition. □

3.4. The new search tree algorithm

In this section, we are going to explain our new search tree algorithm for (Annotated) Dominating Set on planar graphs. In order to be able to conclude our stated running times, we in fact need a corollary of Theorem 3.1 first:

Corollary 3.2. *Let G be a cautiously reduced planar black and white graph. Then, G contains a black vertex of degree at most* 7.

Proof. Let G' be the graph obtained when reducing G further with respect to all reduction rules (R1)–(R7). In particular, each connected component of G' is nearly reduced. Hence, there exists a black vertex v with $\deg_{G'}(v) \leqslant 7$ (in one such component). The only difference between G' and G is that G may contain white vertices of degree two where both neighbors have more than seven neighbors that are of degree at least 4. We argue that $\deg_G(v) \leqslant 7$. If this were not the case, then v must have additional neighbors which are not present in G'. By the above observation an additional neighbor must be a white vertex u of degree two where both neighbors (in particular, the neighbor v) have more than seven neighbors that are of degree at least 4. Hence, there exist vertices $v_1, \ldots, v_\ell \in N_G(v)$ ($\ell \geqslant 8$) which are of degree at least 4. Since these vertices are not removed by any of the reduction rules, it follows that $v_1, \ldots, v_\ell \in N_{G'}(v)$ which implies $\deg_{G'}(v) > 7$, a contradiction. □

Theorem 3.3. (Annotated) Dominating Set *on planar graphs can be solved in $O(8^k n)$ time.*

Proof. Use Corollary 3.2 for the construction of a search tree as described in the introduction given in Section 1. This gives the following algorithm, initiated with the call `pds-st(`$V, \emptyset, E, k, \emptyset$`)`, where $((V, E), k)$ is the given planar graph instance.

```
pds-st(B, W, E, k, S):
  // B is the set of black vertices of the graph instance
  // W is the set of white vertices of the graph instance
  // E is the set of edges of the graph instance
  // k is the parameter of the instance
  // S is the partial solution ''found'' so far

  // preprocessing
  Exhaustively apply ''cautious reduction rules'' to (B, W, k);
  IF k = 0 AND B = W = Ø THEN return S;
  IF k = 0 AND (B ≠ Ø OR W ≠ Ø) THEN return Ø;

  // branching if k > 0
  pick some black vertex v of minimum degree;
  B' := B ∩ N[v];
  W' := W ∩ N[v];
  FOREACH v' ∈ B' DO
     E' := {{u, v'} | u ∈ B ∪ W};
     S' := pds-st (B \ N[v'], W ∪ N(v'), E \ E', k − 1, S ∪ {v'});
     IF S' ≠ Ø THEN break;
  OD;
  IF S' = Ø THEN
     FOREACH v' ∈ W' DO
        E' := {{u, v'} | u ∈ B ∪ W};
        S' := pds-st(B \ N(v'), (W ∪ N(v')) \ {v'}, E \ E', k − 1, S ∪ {v'});
        IF S' ≠ Ø THEN break;
     OD;
  return S
```

Note that performing the reduction in each node of the search tree, by Lemma 3, can be done in time $O(n)$. Moreover, it would be also possible to incorporate reduction rule (R3) to avoid further recursive calls; the time analysis is valid in this case, as well. □

Alternatively, using a reduction to a linear size problem kernel for Dominating Set on planar graphs shown in [4], we obtain the following result.

Theorem 3.4. (Annotated) Dominating Set *on planar graphs can be solved in* $O(8^k k + n^3)$ *time.*

Proof. Use the same search tree algorithm as in Theorem 3.3, just doing an additional preprocessing that computes a size $O(k)$ problem kernel planar graph (actually an instance of Annotated Dominating Set) in $O(n^3)$ time [4]. □

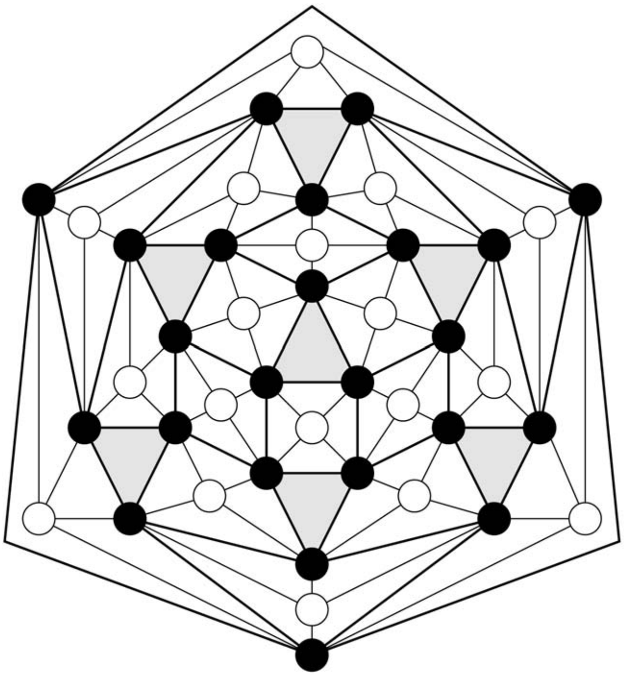

Fig. 2. A graph that shows optimality of the bound derived in our branching theorem.

3.5. Optimality of the branching theorem

We conclude this section by the observation that, with respect to the set of reduction rules we introduced in Section 3.1, the upper bound in our branching theorem is optimal. More precisely, there exists a plane reduced black and white graph with the property that all black vertices have degree 7. Such a graph is shown in Fig. 2. Moreover, this example can be generalized to an infinite family $\mathcal{G}$ of plane reduced black and white graphs with the property that all black vertices have degree 7. The example given in Fig. 2 is the smallest of all graphs in this class.

Each of the graphs in $\mathcal{G}$ could be imagined to be drawn on a can or, mathematically speaking, on a cylinder. On the bottom and the top of the cylinder, we embed the graph depicted in Fig. 3. The vertices with numbers 1 through 9 are at the rim of the top or of the bottom of the can. These numbers are meant as an "interface" to the surface wrapped around the side face of the can. The (general) graph pattern used on the side face is depicted in Fig. 4. It consists of two types of horizontal stripes. If the upper one is denoted by $S_\square$ and the lower one by $S_\triangle$, then consider some sidewall with the pattern described by the expression $S_\triangle(S_\square S_\triangle)^n$ for some $n \geqslant 0$. Hereby, the upper row of black vertices in the uppermost stripe of the type $S_\triangle$ is numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 1. This describes the "can graph" G_n. The graph G_n has

$$2 * 9 * n \;\; \text{[the side wall]} + 2 * 12 \;\; \text{[the top and bottom]} \;\; = 18n + 24$$

black vertices (each of degree seven) and

$$15 * n + 6 \;\; \text{[the side wall]} + 2 * 6 \;\; \text{[the top and bottom]} \;\; = 15n + 18$$

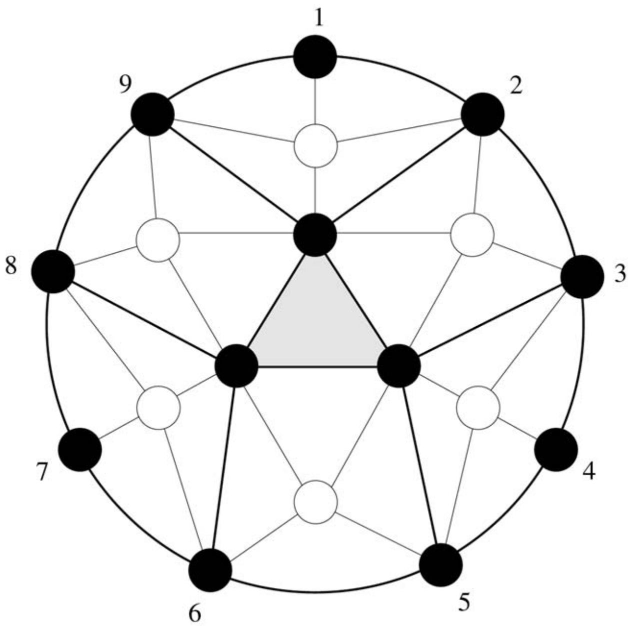

Fig. 3. The top and bottom of the sample can.

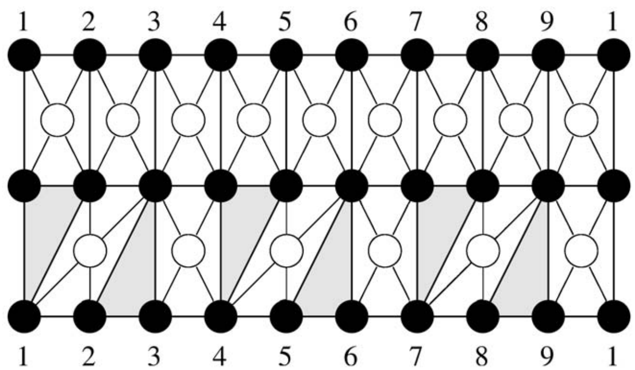

Fig. 4. The sidewall pattern of the sample can.

white vertices (each of degree four). As the reader may verify, G_0 is the graph depicted in Fig. 2. Moreover, none of the graphs G_n is reducible by means of any of the rules listed in Section 3.1.

It is an interesting and challenging task to ask for further reduction rules that would yield a provably better constant in the branching theorem. For example, one might think of the following straightforward generalization of reduction rule (R6):

(R6″) If there are white vertices $u_1, u_2 \in W$ with $N_G(u_1) \subseteq N_G(u_2)$, then delete u_1.

However, the graph in Fig. 2 is reduced even with respect to this generalized rule (R6″). Note that it is not clear how to carry out this reduction rule in linear time (as is also the case for the original rule (R6)).

4. Conclusion

In this paper, we have given the first search tree algorithm proven to be correct (in particular, yielding fixed-parameter tractability) for the Dominating Set problem on planar graphs. Our result improves on the original, flawed claims in [14,15] stating an exponential running time term of 11^k, that we improve here to 8^k.

Let us mention that the *algorithm* which we propose for Dominating Set on planar graphs is close to trivial, while its analysis is very tricky but mathematically pleasing, since it offers new structural insights. Conversely, all advanced search tree algorithms we are aware of (most notably for Vertex Cover [8,16,25,27]) are quite tricky, although their analysis is a rather straightforward but tedious case analysis which is not very elegant from a mathematical point of view.

The proof of our results for the search tree are ultimately based on the Euler formula. A generalization to the class of graphs $\mathcal{G}(S_g)$ (allowing a crossing-free embedding on an orientable surface S_g of genus g) is given in [17]. Other recent considerations (not employing search trees) concerning the investigation of Dominating Set on generalizations of planar graphs can be found in [9–12,20,21].

The proof of our results relied heavily on the presented reduction rules. Recently, it has been empirically shown that a combination of the reduction rules presented here with the reduction rules presented in [4] (which led to a linear size problem kernel) results in a useful algorithm to provide exact solutions for domination problems on large sparse (not necessarily planar) graphs (with up to several thousands of vertices) [2]. In particular, the reduction rules were tested on graphs that are related to the structure of the Internet [2]. It was concluded in [2] that these reduction rules should always be for preprocessing in algorithms that search for high-quality solutions for domination problems. In general, the discovery of powerful preprocessing rules might be viewed as one of the central productive outcomes of research in parameterized algorithms.

An immediate open question deriving from our work is whether one can improve the branching theorem by adding further, more involved reduction rules besides the ones given here and in [2]. Also, it would be interesting to investigate whether and how the algorithm presented here might be combined with the technically more intricate ones based on tree and branch decompositions [3,19]. A broader view on providing exact algorithms for hard problems on planar graphs (together with some experimental findings) can be found in [1].

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